

MITCHELL-STYLE FORCING, WITH SMALL WORKING PARTS AND COLLECTIONS OF MODELS AS SIDE CONDITIONS, AND GAP-ONE SIMPLIFIED MORASSES

CHARLES MORGAN

ABSTRACT. We give a modification of Mitchell’s technique ([8]-[11]) for adding objects of size ω_2 with conditions with finite working parts in which the collections of models used as side conditions are very highly structured, arguably making them more wieldy. We use one such forcing (essentially a ‘pure side conditions’ forcing) to answer affirmatively the question, asked independently by Shelah and Velleman in the late 1980s, as to whether a $(\kappa^+, 1)$ -simplified morass can be added by a forcing with working parts of size $< \kappa$.

INTRODUCTION

The technique of *forcing* is a way of moving from one collection/universe of sets/model of set theory to another. Each forcing allows one to pass from any collection of sets possessing the properties required to fuel the forcing to a new one. The destination model will share many of the features of the point of departure, including satisfying the same basic axioms (usually ZFC or ZF) and having the same ordinals, but will also have some differences.

Forcings are typically designed with specific desired features of the destination models in mind. While the starting point either may not or assuredly will not have these features, the destination model is guaranteed to have them. On the other hand, one usually wants to control at the same time whether, or the extent to which, other important features of the original model are lost on the voyage; *being a cardinal* is a prominent example.

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The constituent elements of many forcings can be thought of informally as each breaking down into a part which gives ‘positive’ information about a desired destination model feature and another which gives constraints, limitations or ‘negative’ information – often aimed at ensuring that key origin model features are not lost. Such descriptions can be useful ways of thinking, albeit that they should not be taken too literally. These ‘positive’ parts have also been called *working parts*.¹ Notwithstanding the fact that everything in an element of a forcing does some work, this evocative designation has gained currency and we use it here.

A fascinating phenomena discovered relatively early in the development of forcing was that some individual desired combinatorial features can be forced into existence by forcings with differing sizes of working part. In particular, some uncountable structures of size ω_1 can be forced not only by using conditions with countable working parts, but also, seemingly less likely intuitively, perhaps, by ones with finite working parts.

An early example is that of adding Souslin trees. Tennenbaum’s original proof using a partial order consisting of finite conditions can be contrasted with Jech’s forcing with countable initial segments. Each forcing has a property that ensures the ω_1 of an origin model remains a cardinal in the corresponding destination model: Tennenbaum’s has the countable chain condition, while Jech’s is countably closed.

Another instructive example is that of adding closed unbounded subsets to stationary subsets of ω_1 . Baumgartner, Harrington and Kleinberg used countable putative initial segments as conditions; the forcing is countably closed. Baumgartner then gave a forcing with finite conditions and which uses *side conditions* to restrict the way in which the working parts can be extended. This partial order does not have the countable chain condition, but cardinals are preserved as the forcing is proper.

Somewhat later it was seen that even certain structures of size ω_2 can sometimes be added by forcing with finite working parts. Ensuring the preservation of cardinals in such arguments, however, is more delicate. Often, as in [17], [1], [24], [13] and later papers, it is necessary to use auxiliary functions definable from structured objects available in the ground model to do so. In fact, some of the most efficacious of these

¹The first use of the phrase ‘working parts’ in this context of which I am aware is in [20].

objects are not available in all ground models and must either themselves be added by forcing or be derivable from the internal structure of the ground model.

Eventually this approach was subsumed by a more powerful one, introduced by Koszmider ([7] – see also [3], [14], [15]). Koszmider’s work adapted Todorćević’s elaboration, in terms of \in -chains of models, of Baumgartner’s idea of using side conditions to restrict the possible extensions of forcing conditions, to the structured objects mentioned above.

A key difference is that in Koszmider’s technique the structured objects are sampled dynamically during the forcings, in contrast to fixed, static uses in the earlier papers. For some time it was reasonable to think that this was the ‘right’ way to do such forcing while avoiding collapsing cardinals by using ‘collections of models as side conditions.’

Mitchell ([8]-[11]) then produced remarkable arguments where the structure on the collection of models, whilst still intricate, was produced as part of the forcing. One needs almost nothing to power the forcing apart from some weak cardinal arithmetic constraints (for example, $2^\omega < \lambda$ when one wants to preserve ω_1 and some larger λ).

These arguments should be seen as being more dynamic again. Not only are the collections of models used as side conditions chosen on the fly, but the structured collection from which they are chosen is also generated spontaneously.

It would be good to understand better the properties of the structured collections added as a by-product of the constructions’ main objectives when using Mitchell’s technique, and to see to what extent those properties rely on the collections being dynamically assembled, and when and whether, in contrast, static axiomatizations of them could be useful.

Mitchell places constraints on the collections of models which form the side conditions which are visually strongly reminiscent of the patterns formed by the ranges of maps in gap-one simplified morasses – see, for example, [9], Diagrams 1 and 2. This resemblance speaks to the prescience of Velleman’s definition of gap-one simplified morasses ([22]). Nevertheless, this paper does not address the relationship between gap-one simplified morasses and the structured collections of models added in the course of Mitchell’s arguments directly.

Instead we show that one can use Mitchell-style forcing arguments to add gap-one simplified morasses themselves. This answers affirmatively the question, asked independently by Shelah and Velleman in the late 1980s, as to whether a $(\kappa^+, 1)$ -simplified morass can be added by a forcing with working parts of size $< \kappa$.

However, rather than use the constraints on collections of models used as side conditions that Mitchell does, here the structure we impose on such collections is a minaturization of that of a gap-one simplified morass.

Hence the paper also shows that one can successfully run Mitchell-style arguments where the structured collections of models added in the course of the construction not only in some ways resembles a gap-one simplified morass, but actually gives rise to one. It seems reasonable to hope that the extensive analysis of gap-one simplified morasses already carried out may prove useful in pushing through further Mitchell-style arguments in which the collections of side conditions, as here, actually give rise to a gap-one simplified morass.

In the forcings discussed so far in this introduction the models in the collections constituting side conditions are all of the same size, for example they will all be countable if the working parts are all finite. In recent work, Neeman and others ([16], [4], [5] and so on) have used forcings with small working parts and collections of models of more than one size as side conditions to prove extensions of the proper forcing axiom and results in cardinal arithmetic. In work in preparation we will build on the results of this paper and discuss relations between such forcings and higher gap simplified morasses.

This paper is self-contained, and no external material concerning simplified morasses is needed. However, there is an extensive literature on gap-one simplified morasses to which the reader looking for background material can refer, including, *inter alia*, [22], [23], [2], [13], [14].

We now give a more concrete outline of the remainder of the paper. We start by mentioning some of the notation that will be used, then remind the reader of the definition of simplified morasses and conclude by sketching the forcing we shall use and summarizing its key properties.

Most of our set theoretic notation is standard.

Notation 1.1. On is the class of ordinals and Card is the collection of infinite cardinals. The letter κ will always denote a cardinal. If X is a set and κ a cardinal we write $[X]^\kappa$ for $\{Y \subseteq X \mid \overline{Y} = \kappa\}$ and $[X]^{<\kappa}$ for $\{Y \subseteq X \mid \overline{Y} < \kappa\}$. If a is a set of ordinals then $\text{ssup}(a)$ is the least ordinal ξ such that $a \subseteq \xi$. For $\alpha, \beta \in \text{On}$ with $\alpha < \beta$ we write $f : \alpha \rightarrow_{\text{o.p.}} \beta$ to indicate that f is an order preserving function from α to β .

We often use the following (standard) notation as short-hand.

Definition 1.2. For $\alpha, \beta \in \text{On}$ with $\alpha < \beta$ we write $(\beta)^\alpha$ for the set $\{f \mid f : \alpha \rightarrow_{\text{o.p.}} \beta\}$ – equivalently the set of increasing sequences of length α from β , and set $(\beta)^{<\alpha} = \bigcup_{\gamma < \alpha} (\beta)^{<\gamma}$.

The following definition is also used frequently.

Definition 1.3. For $\theta, \phi \in \text{On}$ with $\tau < \theta$ we say $\mathcal{F} = \{\text{id}, h\}, \subseteq (\theta)^\tau$, is an *amalgamation pair* if there is some $\sigma < \tau$ such that $h \upharpoonright \sigma = \text{id}$, for all ξ such that $\sigma + \xi < \tau$ we have $h(\sigma + \xi) = \tau + \xi$, and $\tau \cup h''\tau$ is an initial segment of ϕ . In this case we say that σ is the *splitting point* of \mathcal{F} . We say \mathcal{F} is *exact* if $\phi = \tau \cup h''\tau$ and *almost exact* if $\phi = (\tau \cup h''\tau) + 1$.

We recall the definition of an $(\kappa^+, 1)$ -simplified morass. Parsing the definition below, a simplified morass consists of an increasing sequence of length κ^+ of ordinals less than κ^+ , with κ^{++} added at the top – the *levels* of the simplified morass – and collections of maps from lower levels to higher ones.

The remaining axioms specify: the possible maps from one level to its successor, that pairs of maps up to limit levels ‘split’, *i.e.*, factor through a common second factor, that maps from one level to another factor at every intermediate level and there there are only at most κ many maps from any level to any other below the top level. Finally, every element of κ^{++} is in the range of some map up to the top level.

Definition 1.4. ([22]) $\langle \langle \theta_\alpha \mid \alpha \leq \kappa^+ \rangle, \langle \mathcal{F}_{\alpha\beta} \mid \alpha \leq \beta \leq \kappa^+ \rangle \rangle$ is an $(\kappa^+, 1)$ -simplified morass if

- $\langle \theta_\alpha \mid i < \kappa^+ \rangle \in (\kappa^+)^{\kappa^+}$ and $\theta_{\kappa^+} = \kappa^{++}$
- for each $\alpha \leq \beta \leq \kappa^+$ one has that $\mathcal{F}_{\alpha\beta} \subseteq \{f \mid f : \theta_\alpha \rightarrow_{\text{o.p.}} \theta_\beta\}$ and

- for each $\alpha \leq \kappa^+$, $\mathcal{F}_{\alpha\alpha} = \{\text{id}\}$
- for each $\alpha < \kappa^+$,
 $\mathcal{F}_{\alpha\alpha+1}$ is a singleton or an amalgamation pair
- for each $\alpha \leq \beta \leq \gamma \leq \kappa^+$,
 $\mathcal{F}_{\alpha\gamma} = \{g \cdot f \mid f \in \mathcal{F}_{\alpha\beta} \text{ \& } g \in \mathcal{F}_{\beta\gamma}\}$
- for each $\alpha \leq \beta < \kappa^+$, $\overline{\mathcal{F}_{\alpha\beta}} < \kappa^+$
- if $\varepsilon \leq \kappa^+$ is a limit ordinal then the simplified morass is *directed at ε* – i.e., if $\alpha, \beta < \varepsilon$, $e_\alpha \in \mathcal{F}_{\alpha\varepsilon}$ and $e_\beta \in \mathcal{F}_{\beta\varepsilon}$ there is some $\gamma \in [\alpha \cup \beta, \varepsilon)$, $g \in \mathcal{F}_{\gamma\varepsilon}$, $f_i \in \mathcal{F}_{\alpha\gamma}$ and $f_j \in \mathcal{F}_{\beta\gamma}$ such that $e_\alpha = g \cdot f_\alpha$ and $e_\beta = g \cdot f_\beta$
- $\bigcup\{f \text{“}\theta_\alpha \mid \alpha < \kappa^+ \text{ \& } f \in \mathcal{F}_{\alpha\kappa^+}\} = \kappa^{++}$.

Let $\lambda > \kappa^+$ be regular. A forcing will be given to add a $(\kappa^+, 1)$ -simplified morass and make $\lambda = \kappa^{++}$ in the generic extension.

Conditions will consist of simplified morass segments of size less than κ , a map into λ (the map ‘up to the top’), which will also eventually be one of the simplified morass maps, and a collection of elementary submodels of H_λ , which combine well with the maps in the simplified morass segment, as side conditions. The simplified morass segments in conditions in a generic for the forcing, having size smaller than κ , do not constitute initial segments of the added simplified morass: stronger conditions interleave more levels.

In section 7 we show that the forcing does indeed add an $(\kappa^+, 1)$ -simplified morass.² This allows us to observe that the forcing must also collapse all cardinals strictly between κ^+ and λ .

The forcing does not have the κ^+ -chain condition, and indeed this is inevitable, because a $(\kappa^+, 1)$ -simplified morass provides one with a counterexample to the weak Chang conjecture and it is well-known that no κ^+ -cc forcing can add such a counterexample.

However the forcing does preserve many cardinals. We survey the background material for these results in the next section, section 2, and in section 4 we show that the forcing is proper, so κ^+ is preserved.

²The simplified morass added by the forcing is not *neat* – although, of course, as is well known, Velleman ([22]) showed how to obtain a neat gap-one simplified morass (one in which $\bigcup\{f \text{“}\theta_0 \mid f \in \mathcal{F}_{0\kappa^+}\} = \kappa^{++}$) from an arbitrary one.

In section 5 we show that λ and all cardinals greater than it are preserved. The programmatic way to do this is to show that the forcing has the variant of properness which preserves λ provided $2^\kappa < \lambda$ and has the λ^+ -chain condition if $2^\kappa \leq \lambda$. We give proofs in this style in order to make clear how such proofs should run in forcing arguments where the forcing conditions discussed here are used as side conditions. However, as the forcing discussed in this paper is almost a pure side conditions forcing, we are actually able to show that it has the λ -chain condition provided $2^\kappa < \lambda$, and thus have from this that λ and all greater cardinals are preserved.

In section 6 we show that if κ is a regular cardinal the forcing is $<\kappa$ -closed and thus κ and all smaller cardinals are preserved. The proof is similar to the proof in section 7, easier in some places, essentially because we are dealing with a descending chain of conditions as opposed to a directed set, but a little more involved in others because of the technical constraints on limit levels of the working parts of the forcing conditions. (Of course, section 6 can be skipped if one is only interested in the case $\kappa = \omega$.)

Finally, in section 8, we collect together the results proven.

2. BACKGROUND MATERIAL ON CARDINAL PRESERVATION AND ‘GOOD’ MODELS.

We start this section by reviewing some relevant definitions and results on cardinal preservation.

Definition 2.1. Let (\mathbb{P}, \leq) be a partial order and $p \in \mathbb{P}$. A set $D \subseteq \mathbb{P}$ is *predense below* p if for all $q \leq p$ there is some $r \in D$ such that there is some $s \in \mathbb{P}$ with $s \leq q, r$.

Definition 2.2. Let χ be a regular cardinal, (\mathbb{P}, \leq) be a partial order and $(\mathbb{P}, \leq) \in N \prec (H_\chi, \in)$. Then $p \in \mathbb{P}$ is (N, \mathbb{P}) -*generic* if for every dense set $D \subseteq \mathbb{P}$ with $D \in N$ we have that $N \cap D$ is predense below p .

The following proposition is essentially from Hyttinen and Rautila’s [6]; Roslanowski and Shelah ([18]) cite it as being due to “folklore; Hyttinen-Rautila.” We give the proof in detail for two reasons. Firstly, the result in [6] only covers preservation of successor cardinals and here we want a mild generalization covering both successor and limit cardinals. Secondly, [6] is interested in iterable forcings whereas here

we only need worry about single step forcings and so can pare away some restrictions on the forcings covered.

Proposition 2.3. (cf.[6], 3.6) *Let \mathbb{P} be a partial order, let λ be an uncountable regular cardinal and let $(2^{\mathbb{P}})^+ \leq \chi$. Suppose that \mathbb{P} is such that for all regular $\mu < \lambda$, there is a set $\mathcal{U} \subseteq \{N \prec (H_\chi, \in) \mid \mu \subseteq N \text{ \& } \overline{N} < \lambda\}$ which is unbounded in $[H_\chi]^{<\lambda}$ and such that for every $p \in \mathbb{P}$ there is some $N \in \mathcal{U}$ with $p \in N$ and some $p^* \leq p$ which is (N, \mathbb{P}) -generic. Then forcing with \mathbb{P} preserves that λ is an uncountable regular cardinal.*

Proof. Let $\mu < \lambda$ be a cardinal. Let $f : \mu \rightarrow \text{On}$ enumerate a set of ordinals in $V[G]$ of size μ . Let $f = \dot{f}^G$ for some \mathbb{P} -name \dot{f} . Let ϕ be the sentence $\forall \kappa < \lambda \forall C \in [\text{ssup}(\text{rge}(\dot{f}))]^\kappa (\text{rge}(\dot{f}) \not\subseteq C)$. Suppose, towards a contradiction, $p \Vdash \phi$. Let $N \in \mathcal{U}$ with $\dot{f}, p, \mathbb{P} \in N$, $\mu \subseteq N$ and $\overline{N} = \kappa < \lambda$.

For each $\alpha < \mu$ let $D_\alpha = \{r \in \mathbb{P} \mid \exists \beta r \Vdash \dot{f}(\alpha) = \beta \text{ or } r \text{ is incompatible with } p\}$. Then for each $\alpha < \mu$ we have that D_α is dense in \mathbb{P} and D_α is definable from \dot{f}, p and \mathbb{P} and hence is an element of N .

For each $\alpha < \kappa$ let $C_\alpha = \{\beta \in \text{On} \mid \exists r \in \mathbb{P} \cap N r \Vdash \dot{f}(\alpha) = \beta\}$, so $C_\alpha \subseteq N$.

By the hypothesis on \mathbb{P} let $p^* \leq p$ be (N, \mathbb{P}) -generic. Then, by the definition of (N, \mathbb{P}) -genericity we have for each $\alpha < \mu$ that $N \cap D_\alpha$ is predense below p^* : if $q \leq p^*$ there is some $r \in N$ and some $\beta \in \text{On}$ such that $r \Vdash \dot{f}(\alpha) = \beta$ and there is some $s \in \mathbb{P}$ with $s \leq q, r$. Hence for all $\alpha < \mu$ and for all $q \leq p^*$ we have that there is some $s \leq q$ such that $s \Vdash \dot{f}(\alpha) \in C_\alpha$ (since r forces this) and thus $p^* \Vdash \dot{f}(\alpha) \in C_\alpha$.

Set $C = \bigcup_{\alpha < \mu} C_\alpha$. It is clear that $C \in V$ and from the previous paragraph we have that $p^* \Vdash \text{rge}(\dot{f}) \subseteq C$. Moreover, as $C_\alpha \subseteq N$ for each $\alpha < \mu$ we have that $\overline{C} \leq \mu \cdot \kappa = \kappa < \lambda$. This gives the sought after contradiction. ▲_{2.3}

Definition 2.4. If \mathbb{P} has the property of the statement of the definition for $\lambda = \omega_1$ it is *proper*. If \mathbb{P} has the property of the statement of the definition for some successor of a regular cardinal $\lambda = \kappa^+$ it is (a mild generalization of being) *κ -proper* in the sense of [6]

We shall also need some technical material on elementary submodels of H_λ for regular $\lambda > \kappa^+$.

Lemma 2.5. *Suppose λ, θ are regular cardinals with $\kappa^+ < \lambda < \theta$. Suppose $(N, \in) \prec (H_\theta, \in)$ and either $\lambda \in N$ or $\text{Card} \cap N$ is unbounded in $\text{Card} \cap \lambda$. Then $(N \cap H_\lambda) \prec H_\lambda$.*

Definition 2.6. A structure (N, \in) is a *good model* if $(N, \in) \prec (H_\lambda, \in)$, $\delta^N = \kappa^+ \cap N < \kappa^+$, $\kappa^+ \in N$ and $\overline{N} = \kappa$.

Lemma 2.7. *If M, N are good models, $M \in N$ and $\text{otp}(M \cap \lambda) < \delta^N$ then $M \cap \lambda \subseteq N$.*

Proof. As $M \in N$ we have $M \cap \text{On} \in N$. Let $\gamma = \text{otp}(M \cap \lambda)$. As $H_\lambda \models \gamma = \text{otp}(M \cap \lambda)$ and $N \prec H_\lambda$ one also has $N \models \gamma = \text{otp}(M \cap \lambda)$. Since $\gamma < \delta$ we have $\text{otp}((M \cap \lambda)^N) = \text{otp}(M \cap \lambda)$, and hence $(M \cap \lambda)^N = \{\alpha \in M \cap \lambda \mid \alpha \in N\} = M \cap \lambda$. ▲_{2.7}

Let λ be a regular cardinal with $2^\kappa \leq \lambda$.

By this assumption, let \triangleleft be a well-ordering of $(\lambda)^{<\kappa^+}$ in order-type λ .

Definition 2.8. (N, \in, \triangleleft) is a *very good model* if (N, \in) is a good model and $(N, \in, \triangleleft) \prec (H_\lambda, \in, \triangleleft)$.

Lemma 2.9. *Suppose (M, \in, \triangleleft) and (N, \in, \triangleleft) are very good models, $M \in N$ and $\text{otp}(M \cap \lambda) < \delta^N$. Then $((\lambda)^{<\kappa^+})^M \subseteq N$.*

(Observe that $((\lambda)^{<\kappa^+})^M = \{f : \gamma \rightarrow_{\text{o.p.}} \lambda \mid \gamma < \delta^M \text{ \& } f \in M\} = \{f : \gamma \rightarrow_{\text{o.p.}} \lambda \mid \gamma < \delta^M \text{ \& } \text{rge}(f) \subseteq M \cap \lambda\}$.)

Proof. By Lemma (2.7) we have $M \cap \lambda \subseteq N$. Let $\langle x_\alpha \mid \alpha < \lambda \rangle$ enumerate $(\lambda)^{<\kappa^+}$. For all $\alpha < \lambda$ we have that $\alpha \in M$ if and only if $x_\alpha \in ((\lambda)^{<\kappa^+})^M$, and similarly for N . Thus $((\lambda)^{<\kappa^+})^M \subseteq ((\lambda)^{<\kappa^+})^N$. ▲_{2.9}

Definition 2.10. If N is a good model write \overline{N} for the transitive collapse of N and let $\pi^N : \overline{N} \rightarrow N$ be the inverse of the transitive collapsing map.

Lemma 2.11. *Suppose N is a good model, $\gamma < \delta^N$, and $g \in (\lambda)^\gamma$ and $\text{rge}(g) \subseteq N \cap \lambda$. Then $g \in N$ if and only if $(\pi^N)^{-1} \cdot g \in \overline{N}$.*

Proof. Immediate. ▲_{2.11}

Definition 2.12. Let $\theta \in \text{On}$ and let $f : \theta \rightarrow \lambda$ be an order-preserving map. A good model (N, \in) *fits* f if and only if $f = \pi^N \upharpoonright \text{On}$. Equivalently, (N, \in) *fits* f if and only if $\text{rge}(f) = N \cap \lambda$.

3. DEFINITION OF THE MAIN FORCING.

Definition 3.1. $S = \langle \langle \theta_i \mid i \leq \zeta \rangle, \langle \mathcal{F}_{ij} \mid i \leq j \leq \zeta \rangle \rangle$ is a *small* $(\kappa^+, 1)$ -*simplified morass segment* (SMS) if

- $\zeta < \kappa$
- $\langle \theta_i \mid i \leq \zeta \rangle \in (\kappa^+)^{\zeta+1}$,
- if $i \leq \zeta$ is a limit ordinal then $\text{cf}(\theta_i) < \kappa$
- for each $i \leq j \leq \zeta$ one has that $\mathcal{F}_{ij} \in [(\theta_j)^{\theta_i}]^{<\kappa}$, and
 - for each $i \leq \zeta$, $\mathcal{F}_{ii} = \{\text{id}\}$
 - for each $i < \zeta$ either
 - $\mathcal{F}_{ii+1} = \{f_i\}$ is a singleton and $\text{ssup}(\text{rge}(f_i)) < \theta_{i+1}$,
or
 - $\mathcal{F}_{ii+1} = \{\text{id}, h_i\}$ is an *almost exact amalgamation pair* (as defined in Definition (1.3))
- if $k \leq \zeta$ is a limit ordinal then
 - for all $i_0, i_1 < k$ and $g_0 \in \mathcal{F}_{i_0 k}$, $g_1 \in \mathcal{F}_{i_1 k}$ there are $j \in [\max(\{i_0, i_1\}), k)$, $h \in \mathcal{F}_{kj}$, $g'_0 \in \mathcal{F}_{i_0 j}$, $g'_1 \in \mathcal{F}_{i_1 j}$ such that $g_0 = h \cdot g'_0$ and $g_1 = h \cdot g'_1$.
- for each $i \leq j \leq k \leq \zeta$ one has that

$$\mathcal{F}_{ik} = \{g \cdot f \mid f \in \mathcal{F}_{ij} \ \& \ g \in \mathcal{F}_{jk}\}$$

For technical reasons (see the remarks in the paragraphs leading up to the definition of S in the proof of Proposition 6.1, ensuring that we can choose the θ_ξ^* suitably) we have defined small SMS in such a way as to make sure that the following lemma holds.

Lemma 3.2. *If S is a small $(\kappa^+, 1)$ -SMS, $i < j \leq \zeta$, and $f \in \mathcal{F}_{ij}$ then f is not cofinal into θ_j .*

Proof. The proof is by induction on j for each fixed i . If $j = i + 1$ the result is immediate from the definition of \mathcal{F}_{ii+1} . (Recall that if $\mathcal{F}_{ii+1} = \{\text{id}, h_i\}$ is an almost exact amalgamation pair then $\text{ssup}(\text{rge}(h_i)) = \theta_i \cup h_i$ “ $\theta_i < \theta_{i+1}$.”) If $j = j' + 1$ for some $j' > i$ then let $f = g \cdot h$ for some $h \in \mathcal{F}_{ij'}$ and $g \in \mathcal{F}_{j'j}$. Again by the definition of $\mathcal{F}_{j'j}$ we have that g is not cofinal and hence neither is f . Finally, if j is a limit

ordinal then choose any $j' \in (i, j)$; then $f = g \cdot h$ for some $h \in \mathcal{F}_{ij'}$ and $g \in \mathcal{F}_{j'j}$. By the inductive hypothesis h is not cofinal, hence neither is f . $\blacktriangle_{3.2}$

Lemma 3.3. *If S is a small $(\kappa^+, 1)$ -SMS, $0 \leq i < \zeta$, $f \in \mathcal{F}_{i+1\zeta}$, $F \in (\lambda)^{\theta_\zeta}$ and N fits $F \cdot f$ then $\mathcal{F}_{ii+1} = \{f_i\}$.*

Proof. If not then $\text{otp}(N \cap \lambda) = \sigma + \xi \cdot 2 + 1$ for some ξ . But this is clearly impossible if $(N, \in) \prec (H_\lambda, \in)$. $\blacktriangle_{3.3}$

Unsurprisingly, a variant of a generalization of the combination of Velleman and Stanley's ubiquitously helpful lemmas for simplified morasses ([22], Lemma (3.2), and [22], Theorem (3.9), respectively) is very useful here.

Lemma 3.4. *Suppose $S = \langle \langle \theta_i \mid i \leq \zeta \rangle, \langle \mathcal{F}_{ij} \mid i \leq j \leq \zeta \rangle \rangle$ is a small $(\kappa^+, 1)$ -SMS, $F \in (\lambda)^{\theta_n}$, $i \leq j \leq \zeta$, $f, g \in \mathcal{F}_{ij}$, and $\xi, \tau \leq \theta_i$. If $\text{ssup}(f \text{``}\xi) = \text{ssup}(g \text{``}\tau)$ or if $j = n$ and $\text{ssup}(F \cdot f \text{``}\xi) = \text{ssup}(F \cdot g \text{``}\tau)$ then $\xi = \tau$ and $f \restriction \xi = g \restriction \xi$.*

Proof. By induction on j for each i . If $j = i + 1$ the lemma is immediate from the two possible cases for the structure of \mathcal{F}_{ii+1} . If $j = i' + 1$ for some $i' > i$, note that one can factor f, g as $f = f_1 \cdot f_0$ and $g = g_1 \cdot g_0$ with $f_0, g_0 \in \mathcal{F}_{ii'}$ and $f_1, g_1 \in \mathcal{F}_{i'i'+1}$. If $\mathcal{F}_{i'i'+1}$ is a singleton then this instance of the lemma is immediate from the lemma for i and i' . Otherwise, $\mathcal{F}_{i'i'+1}$ is an amalgamation pair with splitting point $\sigma_{i'}$ and either $f_1 = g_1$, in which case the lemma is again immediate by the induction hypothesis, or $f_1 \neq g_1$ and hence $\text{ssup}(f_0 \text{``}\xi) = \text{ssup}(g_0 \text{``}\tau) \leq \sigma_{i'}$ and so the lemma is once again true. If $j \leq \zeta$ is a limit ordinal then there is some $k \in [i, j)$, $f' \in \mathcal{F}_{ik}$, $g' \in \mathcal{F}_{ik}$ and $h \in \mathcal{F}_{kj}$ such that $f = h \cdot f'$ and $g = h \cdot g'$. If $\text{ssup}(f \text{``}\xi) = \text{ssup}(g \text{``}\tau)$ then $\text{ssup}(f' \text{``}\xi) = \text{ssup}(g' \text{``}\tau)$. By the inductive hypothesis the lemma is true for k, f' and g' . Hence, applying h on both sides, it is true of j, f and g . If $j = \zeta$ then $\text{ssup}(F \cdot f \text{``}\xi) = \text{ssup}(F \cdot g \text{``}\tau)$ if and only if $\text{ssup}(f \text{``}\xi) = \text{ssup}(g \text{``}\tau)$, and so the lemma for $F \cdot f$ and $F \cdot g$ is true by the lemma for f and g . $\blacktriangle_{3.4}$

We now introduce the main forcing notion that will be considered in this paper.

Definition 3.5. \mathbb{P} is the forcing notion in which conditions other than $\mathbb{1}$ are of the form $p = (S^p, \mathcal{F}^p, \mathcal{N}^p)$ where

- $S^p = \langle \langle \theta_i^p \mid i \leq \zeta^p \rangle, \langle \mathcal{F}_{ij}^p \mid i \leq j \leq \zeta^p \rangle \rangle$ is a small $(\kappa^+, 1)$ -SMS,
- $\mathcal{F}^p = \{(F^p \cdot f, \theta_i^p) \mid i \leq \zeta^p \text{ \& } f \in \mathcal{F}_{i\zeta^p}\}$ for some $F^p \in (\lambda)^{\theta_{n^p}^p}$,
- \mathcal{N}^p is a set of very good models such that for every $M \in \mathcal{N}^p$ there is a unique $(i^M, f^M) \in \bigcup_{j \leq \zeta^p} \{j\} \times \mathcal{F}_{j\zeta^p}^p$ such that M fits $F^p \cdot f^M$ and
 - $\circ_1 \ \forall i \leq j \leq i^M \ (\mathcal{F}_{ij}^p \subseteq \overline{M})$
 - $\circ_2 \ \forall N \in \mathcal{N}^p \ \forall \gamma < \delta^N \ \forall g \in (\theta_{i^N})^\gamma$

$$(i^N \leq i^M \ \& \ g \in \overline{N} \implies g \in \overline{M})$$

We sometimes write instead, more informally, $p = (S^p, F^p, \mathcal{N}^p)$, as clearly \mathcal{F}^p is recoverable from F^p and S^p .

We take $\mathbb{1}$ to be the empty condition $(\emptyset, \emptyset, \emptyset)$ and, for numerological reasons, formally take $\zeta^{\mathbb{1}} = -1$.

The ordering is given by $q \leq p$ if $(p = \mathbb{1} \text{ or})$

- $\zeta^p \leq \zeta^q$,
- there is some $k : \zeta^p + 1 \longrightarrow_{\text{o.p.}} \zeta^q + 1$ such that
 - for all $i \leq \zeta^p$ one has $\theta_i^p = \theta_{k(i)}^q$,
 - for all $i \leq j \leq \zeta^p$ one has $\mathcal{F}_{ij}^p \subseteq \mathcal{F}_{k(i)k(j)}^q$,
 - if $k(i+1) = k(i) + 1$ then $\mathcal{F}_{ii+1}^p = \mathcal{F}_{k(i)k(i)+1}^q$,
- there is some $f^{pq} \in \mathcal{F}_{k(\zeta^p)\zeta^q}^q$ such that $F^p = F^q \cdot f^{pq}$,
- $\mathcal{N}^p \subseteq \mathcal{N}^q$,
- if $N \in \mathcal{N}^q$ is witnessed by $(k(i), f^{pq} \cdot g)$ for some $g \in \mathcal{F}_{i\zeta^p}^p$, then $N \in \mathcal{N}^p$ (and this is witnessed by (i, g) since $F^q \cdot f^{pq} \cdot g = F^p \cdot g$).

The clauses \circ_1 and \circ_2 are succinct and straightforward. However it is useful, for the proofs below, to unpack them into a sequence of conditions which involve the small SMS structure more explicitly.

Lemma 3.6. *Suppose (as in Definition (3.5)) that $p = (S^p, \mathcal{F}^p, \mathcal{N}^p)$, where $S^p = \langle \langle \theta_i^p \mid i \leq \zeta^p \rangle, \langle \mathcal{F}_{ij}^p \mid i \leq j \leq \zeta^p \rangle \rangle$ is a small $(\kappa^+, 1)$ -SMS, $\mathcal{F}^p = \{(F^p \cdot f, \theta_i^p) \mid i \leq \zeta^p \text{ \& } f \in \mathcal{F}_{i\zeta^p}\}$ for some $F^p : \theta_{\zeta^p}^p \longrightarrow_{\text{o.p.}} \lambda$, and*

\mathcal{N}^p is a collection of very good models such that for all $M \in \mathcal{N}^p$ there is some (unique) $(i^M, f^M) \in \bigcup_{j \leq \zeta^p} \{j\} \times \mathcal{F}_{j\zeta^p}^p$ such that M fits $F^p \cdot f^M$.

Then $p \in \mathbb{P}$, i.e., \circ_1 and \circ_2 hold, if and only if the following conditions hold:

- ₁ $\forall i \leq j < i^M (\theta_i^p < \delta^M \ \& \ \mathcal{F}_{ij}^p \subseteq M)$
- ₂ $\forall i < i^M \forall g \in \mathcal{F}_{ii^M} (F^p \cdot f^M \cdot g \in M)$
- ₃ $\forall N \in \mathcal{N}^p (i^N \leq i^M \implies ((\kappa^+)^{<\kappa^+})^N \subseteq ((\kappa^+)^{<\kappa^+})^M)$
- ₄ $\forall N \in \mathcal{N}^p \forall \gamma < \delta^N \forall g \in (\theta_{i^N})^\gamma (i^N = i^M \implies (F^p \cdot f^N \cdot g \in N \iff F^p \cdot f^M \cdot g \in M))$
- ₅ $\forall N \in \mathcal{N}^p \forall \gamma < \delta^N \forall g \in (\theta_{i^N})^\gamma (i^N < i^M \implies (F^p \cdot f^N \cdot g \in N \implies g \in M))$

Proof. For $M \in \mathcal{N}^p$ we have $\pi^M \restriction \text{On} = F^p \cdot f^M$. Note that $\pi^M \restriction \delta^M = \text{id}$, $\pi^M(\delta^M) = \kappa^+$ and that $\forall i < i^M \theta_i < \delta^M$. Observe that it is now immediate that for $i \in \{1, 2, 3, 4, 5\}$ we have that \bullet_i holds if and only if \bullet'_i does, where the \bullet'_i are given by

- ₁' $\forall i \leq j < i^M \mathcal{F}_{ij}^p \subseteq \overline{M}$
- ₂' $\forall i < i^M \mathcal{F}_{ii^M}^p \subseteq \overline{M}$
- ₃' $\forall N \in \mathcal{N}^p \forall \gamma < \delta^N \forall g \in (\delta^N)^\gamma (i^N \leq i^M \implies (g \in \overline{N} \iff g \in \overline{M}))$
- ₄' $\forall N \in \mathcal{N}^p \forall \gamma < \delta^N \forall g \in (\theta_{i^N})^\gamma (i^N = i^M \implies (g \in \overline{N} \iff g \in \overline{M}))$
- ₅' $\forall N \in \mathcal{N}^p \forall \gamma < \delta^N \forall g \in (\theta_{i^N})^\gamma (i^N < i^M \implies (g \in \overline{N} \implies g \in \overline{M}))$

The conclusion now follows as \circ_1 is the conjunction of \bullet'_1 and \bullet'_2 and \circ_2 is the conjunction on \bullet'_3 , \bullet'_4 and \bullet'_5 . ▲_{3.6}

Before proceeding we give a useful lemma showing that for a condition $p \in \mathbb{P}$ the relationship of one member of the collection of good models \mathcal{N}^p being an element of another behaves well with respect to the SMS structure S^p of p .

Lemma 3.7. *Suppose $p = (M, \mathcal{F}, \mathcal{N}) \in \mathbb{P}$ and write ζ for ζ^p . If $N, N' \in \mathcal{N}$ with $i_{N'} < i_N$ and $N' \in N$ then $\exists f \in \mathcal{F}_{i_{N'}i_N} (f_{N'} = f_N \cdot f)$.*

Proof. By \bullet_1 for p applied to $\theta_{i_{N'}}$ and N one has $\text{otp}(N' \cap \lambda) < \delta^N$. By hypothesis $N' \in N$, so, by Lemma (1.7), $\text{rge}(F \cdot f_{N'}) = N' \cap \lambda \subseteq N \cap \lambda = \text{rge}(F \cdot f_N)$. As F is order preserving $\text{rge}(f_{N'}) \subseteq \text{rge}(f_N)$. By the last property of $\mathcal{F}_{i_{N'}\zeta}$ given in Definition (3.5), we may factor $f_{N'}$ as $h \cdot f$ where $h \in \mathcal{F}_{i_N\zeta}$ and $f \in \mathcal{F}_{i_{N'}i_N}$. Then $h \upharpoonright \text{ssup}(\text{rge}(f)) = f_N \upharpoonright \text{ssup}(\text{rge}(f))$, by Lemma (3.4), so $f_{N'} = f_N \cdot f$. $\blacktriangle_{3.7}$

4. PRESERVATION OF κ^+ .

Proposition 4.1. *\mathbb{P} is proper and hence forcing with it preserves κ^+ .*

Proof. Let $x \in H_\chi$ and $p \in \mathbb{P}$. Suppose N is a very good model with $x, p \in N$, $\{g \in (\theta_{i_M}^p)^\gamma \mid M \in \mathcal{N}^p \ \& \ \gamma < \delta^M \ \& \ F^p \cdot f^M \cdot g \in M\} \subseteq N$, and $\text{cf}(N \cap \lambda) = \kappa$. (It is easy to find such an N by the downward Lowenheim-Skolem theorem.)

As $\theta_{\zeta^p}^p \in N \cap \kappa^+$ one has that $\theta_{\zeta^p}^p \subseteq N$. Since $F^p \in N$ we have that $F^p \restriction \theta_{\zeta^p}^p \subseteq N$.

Let $\theta^* = \text{otp}(N \cap \lambda)$, let F^* be the inverse of the transitive collapse of $N \cap \lambda$ and let $f^* = F^{*-1} \cdot F^p$. Now define $p^* = (S^*, \mathcal{F}^*, \mathcal{N}^*)$ by $S^* = \langle \langle \theta_i^p \mid i \leq \zeta^p \rangle \cap \theta^*, \langle \mathcal{F}_{ij}^p \mid i \leq j \leq \zeta^p \rangle \cap \langle \{f^* \cdot f \mid f \in \mathcal{F}_{i\zeta^p}^p\} \mid i \leq \zeta^p \rangle \rangle$, $\mathcal{F}^* = \{(F^* \cdot f, \theta_i^*) \mid i \leq \zeta^p + 1 \ \& \ f \in \mathcal{F}_{i\zeta^p+1}^*\}$ and $\mathcal{N}^* = \mathcal{N} \cup \{N\}$.

Lemma 4.2. $p^* \in \mathbb{P}$

Proof. The witnessing pair for N is (ζ^{p^*}, id) . The only thing to check is that N satisfies $\bullet_1 - \bullet_5$.

\bullet_1 : It is immediate that $\mathcal{F}_{ij}^* \subseteq N$ for all $i \leq j \leq \zeta^p$ as $\mathcal{F}_{ij}^* = \mathcal{F}_{ij}^p$ for all $i \leq j \leq \zeta^p$ and $\mathcal{F}_{ij}^p \subseteq N$ since $p \in N$, ζ^p and each \mathcal{F}_{ij}^p has size $< \kappa$ and ${}^{<\kappa}N \subseteq N$.

\bullet_2 : It is also immediate that $F^* \cdot f^* \cdot g, = F^p \cdot g, \in N$ for all $g \in \mathcal{F}_{i\zeta^p}$ as for each $g \in \mathcal{F}_{i\zeta^p}$ one has $g \in N$, as mentioned in the proof of \bullet_1 , and $F^p \in N$ (since $\mathcal{F}^p \in N$ has size $< \kappa$ and ${}^{<\kappa}N \subseteq N$, and $F^p = F^p \cdot \text{id} \in \mathcal{F}^p$).

•₃ - •₅: Lemma (2.9) gives •₃, •₄ is vacuously true, and •₅ is true by the construction of N . ▲_{4.2}

Lemma 4.3. p^* is (N, \mathbb{P}) -generic.

Proof. Let $q \leq p^*$, and let $m^* \leq \zeta^q$ be such that $\theta_{m^*}^q = \theta^*$.

By the definition of small SMS we have two cases: $m^* = 0$ or m^* is a successor ordinal. For if m^* were to be a limit ordinal we would have that $\text{cf}(\theta^*) < \kappa$, contradicting the fact that $\text{cf}(N \cap \lambda) = \kappa$.

In the former case we take $q \restriction N = \mathbb{1}$. Clearly in the case $q \restriction N \in \mathbb{P}$.

Now suppose (until the end of Claim (4.4)) that we are in the other case, so that $m^* = m + 1$ for some $m < \zeta^q$. In this case $\theta_{m+1}^q = \theta^*$. Note, as remarked in Lemma (3.3), that we must have that \mathcal{F}_{mm+1}^q is the singleton $\{f_m^q\}$.

Define $q \restriction N$ as follows. Let $S^{q \restriction N} = \langle \langle \theta_i^q \mid i \leq m \rangle, \langle \mathcal{F}_{ij}^q \mid i \leq j \leq m \rangle \rangle$. Let $F^{q \restriction N} = F^* \cdot f_m^q$ and $\mathcal{F}^{q \restriction N} = \{(F^* \cdot f_m^q \cdot f, \theta_i^q) \mid i \leq m \text{ \& } f \in \mathcal{F}_{im}^q\}$. Let $\mathcal{N}^{q \restriction N} = \mathcal{N}^q \cap N$. Set $q \restriction N = (S^{q \restriction N}, \mathcal{F}^{q \restriction N}, \mathcal{N}^{q \restriction N})$.

Claim 4.4. $q \restriction N \in \mathbb{P}$.

Proof. We start by showing that the ‘ \mathcal{F} ’-component of $q \restriction N$ is exactly the intersection of the ‘ \mathcal{F} ’-component of q with N .

Subclaim 4.5. $\mathcal{F}^{q \restriction N} = \mathcal{F}^q \cap N$.

Proof. In one direction, if $i \leq m$ and $f \in \mathcal{F}_{im}^q$ one has that

$$F^{q \restriction N} \cdot f = F^* \cdot f_m^q \cdot f = F^q \cdot f^{p^*q} \cdot f_m^q \cdot f,$$

where $(m+1, f^{p^*q})$ witnesses that $N \in \mathcal{N}^q$. Consequently $F^{q \restriction N} \cdot f$ is of the form $F^q \cdot g$ for some $g \in \mathcal{F}_{i\zeta^q}^q$ and $F^{q \restriction N} \cdot f \in N$ (by •₂ for N and q). Thus $\mathcal{F}^{q \restriction N} \subseteq \mathcal{F}^q \cap N$.

In the other direction, suppose $i \leq \zeta^q$, $g \in \mathcal{F}_{i\zeta^q}^q$ and $(F^q \cdot g, \theta_i) \in N$. Then $i \leq m$, since $\theta_i \in N$, and so $F^q \cdot g \restriction \theta_i \subseteq N$. Write g as $g = l \cdot f_m^q \cdot f$ where $f \in \mathcal{F}_{im}^q$ and $l \in \mathcal{F}_{m+1\zeta^q}^q$. Then $F^q \cdot l \restriction (f_m^q \cdot f \restriction \theta_i^q) = F^q \cdot f_N \restriction (f_m^q \cdot f \restriction \theta_i^q)$, where $(m+1, f_N)$ witnesses that $N \in \mathcal{N}^q$ (and hence f_N is the map f^{p^*q}), by Lemma (1.10). Thus $F^q \cdot g = F^q \cdot f^{p^*q} \cdot f_m^q \cdot f = F^* \cdot f_m^q \cdot f = F^{q \restriction N} \cdot f$. Hence $\mathcal{F}^q \cap N \subseteq \mathcal{F}^{q \restriction N}$. ▲_{4.5}

Suppose $N' \in \mathcal{N}^q \cap N$. Then, by Lemma (2.6), $f_{N'} = f_N \cdot f_m^q \cdot f$ for some $f \in \mathcal{F}_{i_{N'}m}^q$. Recalling that $f_N = f^{p^*q}$ and that $F^{q \restriction N} = F^q \cdot f^{p^*q} \cdot f_m^q$, one

thus has that $F^q \cdot f_{N'} = F^{q \upharpoonright N} \cdot f$. Hence (i, f) is the (unique) witness that $N' \in \mathcal{N}^{q \upharpoonright N}$. Consequently \bullet_1 and \bullet_2 for $q \upharpoonright N$ are immediate from these properties for q . Clearly, $\bullet_3 - \bullet_5$ hold since they do for q .

Thus we have shown that $q \upharpoonright N \in \mathbb{P}$.

▲_{4.4}

Now let $s = (S^s, \mathcal{F}^s, \mathcal{N}^s) \in N \cap \mathbb{P}$ be such that $s \leq q \upharpoonright N$.

Claim 4.6. *There is some $r \in \mathbb{P}$ with $r \leq q, s$.*

Proof. Let $m = \zeta^{q \upharpoonright N}$. Let $k : m + 1 \rightarrow \zeta^s + 1$ witness that $s \leq q \upharpoonright N$. Extend k so that $\text{dom}(k) = \zeta^q + 1$ by setting $k(m + 1 + i) = \zeta^s + 1 + i$ for $i \in \text{otp}((m, \zeta^q])$.

Set $\zeta^r = \zeta^s + \text{otp}((m, \zeta^q])$ and $\theta_i^r = \theta_i^s$ for $i \leq \zeta^s$ and $\theta_{\zeta^s+1+i}^r = \theta_{m+1+i}^q$ for $i \in \text{otp}((m, \zeta^q])$. The sequence $\langle \theta_i^r \mid i < \zeta^r \rangle$ is increasing because $\theta_{\zeta^s} \in N$ and hence $\theta_{\zeta^s} < \delta^N < \text{otp}(N \cap \lambda) = \theta_{m+1}^q$.

Let $F^q \cdot f_N$ be the inverse of the transitive collapse of $\text{otp}(N \cap \lambda)$, where $(m + 1, f_N) \in \mathcal{F}_{m+1\zeta^q}^q$ witnesses that $N \in \mathcal{N}^q$.

Again, as $s \in N$ one has that $F^s \in N$ and hence $F^s \text{``} \theta_{\zeta^s} \subseteq N$. Thus one can make the following definition.

For $i \leq j \leq \zeta^s$ let $\mathcal{F}_{ij}^r = \mathcal{F}_{ij}^s$. Let $\mathcal{F}_{\zeta^s \zeta^s+1}^r = \{f_N^{-1} \cdot (F^q)^{-1} \cdot F^s\}$. Hence, in the notation for small SMSs from Definition (3.1), $f_{\zeta^s}^r = f_N^{-1} \cdot (F^q)^{-1} \cdot F^s$. For $1 \leq i \leq j \leq \text{otp}((m, \zeta^q])$ let $\mathcal{F}_{\zeta^s+i\zeta^s+j}^r = \mathcal{F}_{ij}^q$. Finally, for $i \leq j \leq \zeta^r$ with $i \leq \zeta^s$ and $j > \zeta^s$ set $\mathcal{F}_{ij}^r = \{f \cdot f_{\zeta^s}^r \cdot g \mid g \in \mathcal{F}_{i\zeta^s}^r \text{ \& } f \in \mathcal{F}_{\zeta^s+1j}^r\}$.

Then $S^r = \langle \langle \theta_i^r \mid i \leq \zeta^r \rangle, \langle \mathcal{F}_{ij}^r \mid i \leq j \leq \zeta^r \rangle \rangle$ is a small SMS.

Next, set $F^r = F^q$ and $\mathcal{F}^r = \{(F^r \cdot f, \theta_i) \mid i \leq \zeta^r \text{ \& } f \in \mathcal{F}_{i\zeta^r}^r\}$.

Lastly, set $\mathcal{N}^r = \mathcal{N}^s \cup \mathcal{N}^q$.

In order to finish the proof we must show that $\bullet_1 - \bullet_5$ hold for r . If $M \in \mathcal{N}^s$ then $\bullet_1 - \bullet_5$ hold for r since they do for s . So suppose $M \in \mathcal{N}^q \setminus \mathcal{N}^s$.

We shall write i^M for $(i^M)^r$, i^N for $(i^N)^r$, and so on, in the remainder of the proof.

\bullet_1 . Let $f \in \mathcal{F}_{ij}^r$ with $i \leq j < i^M$.

Case 1(i). $i \leq j < i^N \leq i^M$. Then $f \in \mathcal{F}_{ij}^s$. As $s \in N$ we have $\mathcal{F}_{ij}^s \in N$ and so $f \in N$ (because both s and \mathcal{F}_{ij}^s are of size $< \kappa$). As

$s \in N$ we have $\theta_j^s < \delta^N$ and so $f \in ((\kappa^+)^{<\kappa^+})^N$. As $i^N \leq i^M$ we have $((\kappa^+)^{<\kappa^+})^N \subseteq ((\kappa^+)^{<\kappa^+})^M$ by \bullet_3 for q . Thus $f \in M$. \blacktriangle (Case 1(i))

Case 1(ii). $i \leq j = i^N < i^M$. In this case we can factor f as $f = f_m \cdot g$ with $g \in \mathcal{F}_{im}^r = \mathcal{F}_{im}^s$ and f_m is the (unique) map in \mathcal{F}_{mm+1}^r . By definition $f_m = (F^r \cdot f^N)^{-1} \cdot F_s$, so $F^r \cdot f^N \cdot f_m = F^s$. Since $s \in N$ and s, \mathcal{F}^s are of size $< \kappa$, we have that $F^s \in N$. As $\theta_{\zeta^s} < \delta^N$, and by the case hypothesis that $i^N < i^M$, we can apply \bullet_5 for q (to N and M), obtaining that $f_m \in M$. However, we also have that $g \in M$ by Case 1(i). Hence $f = f_m \cdot g \in M$. \blacktriangle (Case 1(ii))

Case 1(iii). $i < i^N < j < i^M$. In this case we can factor f as $f = h \cdot g$ with $g \in \mathcal{F}_{ii^N}^r$ and $h \in \mathcal{F}_{i^N j}^r$. Then $h \in M$ by \bullet_1 for q and $g \in M$ by Case 1(ii). Hence $f \in M$. \blacktriangle (Case 1(iii))

Case 1(iv) $i^N \leq i \leq j < i^M$. In this case $\mathcal{F}_{ij}^r = \mathcal{F}_{ij}^q$ and so $f \in M$ by \bullet_1 for q . \blacktriangle (Case 1(iv)), (Verification of \bullet_1)

\bullet_2 . Let $i < i^M$ and $f \in \mathcal{F}_{ii^M}^r$.

Case 2(i). $i^N \leq i$. Then \bullet_2 for f and M holds by \bullet_2 for q . \blacktriangle (Case 2(i))

Case 2(ii). $i < i^N < i^M$. Factor f as $f = h \cdot f_m \cdot g$ where $h \in \mathcal{F}_{i^N i^M}^r$ and $g \in \mathcal{F}_{im}^r$. Then $F^r \cdot f^M \cdot h \in M$ by \bullet_2 for q . Also, $f_m \cdot g \in M$ by \bullet_1 for r . So $F^r \cdot f^M \cdot h \cdot f_m \cdot g = F^r \cdot f^M \cdot f \in M$. \blacktriangle (Case 2(ii))

Case 2(iii). $i < i^N = i^M$. Factor f as $f = f_m \cdot g$ where $g \in \mathcal{F}_{im}^r$. As in Case 2(ii), $g \in M$ by \bullet_1 for r . As in the proof of Case 1(ii), $F^r \cdot f^N \cdot f_m = F^s \in N$. So $F^r \cdot f^M \cdot f_m \in M$ by \bullet_4 for q . \blacktriangle (Case 2(iii)), (Verification of \bullet_2)

Now let $K \in \mathcal{N}^r$. If $K \in \mathcal{N}^q$ then each of $\bullet_1 - \bullet_5$ holds for r by the corresponding property for q . So suppose $K \in \mathcal{N}^s \setminus \mathcal{N}^q$.

\bullet_3 . By Lemma (2.9) we have that $((\lambda)^{<\kappa^+})^K \subseteq N$. By \bullet_3 for q we have $((\kappa^+)^{<\kappa^+})^N \subseteq ((\kappa^+)^{<\kappa^+})^M$. Thus $((\kappa^+)^{<\kappa^+})^K \subseteq ((\kappa^+)^{<\kappa^+})^M$. \blacktriangle (Verification of \bullet_3)

\bullet_4 . As $i^K < i^M$ there is nothing to check here. \blacktriangle (Verification of \bullet_4)

\bullet_5 . Suppose $\gamma < \delta^K$, $g \in (\theta_{i^K})^\gamma$ and $F^r \cdot f^K \cdot g \in K$. By Lemma (2.9), since s, \mathcal{N}^s and K are all elements of N , we have $((\lambda)^{<\kappa^+})^K \subseteq N$, and so $F^r \cdot f^K \cdot g \in N$.

We also have that $f^K = f^N \cdot f_m \cdot (f^K)^s$, so $F^r \cdot f^K = F^r \cdot f^N \cdot (f_m \cdot (f^K)^s)$. Thus $F^r \cdot f^K \in N$ by \bullet_2 for r applied to N and $f_m \cdot (f^K)^s$.

Hence $g, = (F^r \cdot f^N)^{-1} \cdot (F^r \cdot f^N \cdot g), \in N$. But then $g \in ((\kappa^+)^{<\kappa^+})^N$ and so $g \in M$ by \bullet_3 for q . \blacktriangle (Verification of \bullet_5)

Thus $r = (S^r, \mathcal{F}^r, \mathcal{N}^r) \in \mathbb{P}$. $\blacktriangle_{4.6}$

Hence we have shown that p^* is N -generic. $\blacktriangle_{4.3}$

Thus forcing with \mathbb{P} preserves κ^+ . $\blacktriangle_{4.1}$

5. PRESERVATION OF λ AND GREATER CARDINALS.

We need with a couple of auxiliary definitions for this section.

Definition 5.1. For $p \in \mathbb{P}$ let $Z^p = \{i \leq \zeta^p \mid \exists M \in \mathcal{N}^p (i = i^M)\}$ and, for $i \in Z^p$, $X_i^p = \{g \in (\theta_{i^M})^\gamma \mid \exists M \in \mathcal{N}^p (i = i^M \ \& \ \gamma < \delta^M \ \& \ g \in \overline{M})\}$.

Note, for each $p \in \mathbb{P}$ we have that $\forall i \leq j \leq \zeta^p (X_j^p \neq \emptyset \implies X_i^p \subseteq X_j^p)$, by \circ_2 , and for $M \in \mathcal{N}^p$ that $X_{i^M}^p = \{g \in (\theta_{i^M})^\gamma \mid \gamma < \delta^M \ \& \ g \in \overline{M}\}$.

For $p \in \mathbb{P}$ and $i \in Z^p$ one has that X_i^p is a subset of $(\kappa^+)^{<\kappa^+}$ of size at most κ .

We start by showing that \mathbb{P} satisfies the criterion of Proposition (2.3) for preserving λ . Referring to that criterion, here for each $p \in \mathbb{P}$ we will be able to take the relevant (N, \mathbb{P}) -generic p^* to be p itself. Because of this, as we will see below, a similar argument allows us to show the stronger conclusion that \mathbb{P} has the λ -chain condition. Nevertheless, we give the more general argument here as a template, because variants of it will be necessary when employing \mathbb{P} as a collection of side conditions in forcings \mathbb{Q} (when the (N, \mathbb{Q}) -generic p^* will in general not always be able to be taken to be the $p \in \mathbb{Q}$ itself) which cannot have the λ -chain condition.

Proposition 5.2. *Suppose $2^\kappa < \lambda$. Then forcing with \mathbb{P} preserves λ .*

Proof. Let $x \in H_\chi$ and $\mu < \lambda$. Suppose that D is a dense set in \mathbb{P} and $x, p, D, \mathbb{P} \in N$ where $N \prec H_\chi$, $\mu \leq \delta^N = N \cap \lambda \in \lambda$ and $\text{cf}(\delta^N) \geq \kappa^+$, $\overline{N} < \lambda$, and $N^\kappa \subseteq N$.

Claim 5.3. *If $q \leq p$ and $q \in D$ there is some $s \in D \cap N$ such that $s \leq p$ and s and q are compatible.*

Proof. As $\theta_{\zeta^q}^q < \kappa^+$ we have that $F^q \text{``}\theta_{\zeta^q}^q$ is of size at most κ , so since $N^\kappa \subseteq N$ we have that $Y = N \cap F^q \text{``}\theta_{\zeta^q}^q \in N$. Also, as $N \cap \lambda \in \lambda$ we have that Y is an initial segment of $F^q \text{``}\theta_{\zeta^q}^q$. Also, again as $N^\kappa \subseteq N$, each X_i^q for $i \in Z^q$ is an element of N .

Let $\phi(x)$ be the formula which is the conjunction of the following sub-formulae: $x \in \mathbb{P}$; $S^x = S^q$; $x \leq p$; Y is an initial segment of $F^x \text{``}\theta_{\zeta^x}^x$; $Z^x = Z^q$; and $\langle X_i^x \mid i \in Z^x \rangle = \langle X_i^q \mid i \in Z^q \rangle$.

Since q witnesses $H_\chi \models \exists x \phi(x)$ there is some $s \in N$ such that $N \models \phi(s)$.

It is easy now to amalgamate the morass parts of s and q - our comments about Y have ensured that we have a head-tail-tail amalgamation which we can use in the morass. In more detail, define r by setting $\zeta^r = \zeta^q + 1$, $S^r \restriction \zeta^r = S^q = S^s$, $\theta_{\zeta^r}^r = \text{otp}(\text{rge}(F^s) \cup \text{rge}(F^q)) + 1$, $\mathcal{F}_{\zeta^q \zeta^r}^r = \{\text{id}, h_{\zeta^q}^r\}$, where the splitting point of $\mathcal{F}_{\zeta^q \zeta^r}^r$, $\sigma_{\zeta^q}^r$, is $\text{otp}(Y)$, $\mathcal{F}_{i \zeta^r}^r = \{f \cdot g \mid f \in \mathcal{F}_{\zeta^q \zeta^r}^r \ \& \ g \in \mathcal{F}_{i \zeta^q}^r\}$ for $i < \zeta^q$, $F^r \restriction \text{otp}(\text{rge}(F^s) \cup \text{rge}(F^q))$ is the inverse of the transitive collapse of $\text{rge}(F^s) \cup \text{rge}(F^q)$, $F^r(\text{otp}(\text{rge}(F^s) \cup \text{rge}(F^q))) = \text{ssup}(\text{rge}(F^s) \cup \text{rge}(F^q))$, and $\mathcal{N}^r = \mathcal{N}^s \cup \mathcal{N}^q$.

Subclaim 5.4. \mathcal{N}^r is such that $r \in \mathbb{P}$, whence r witnesses that s and q are compatible.

Proof. The property \circ_1 holds for r immediately from \circ_1 for q and s . We will be finished if we show that \circ_2 also holds for r .

Suppose that $M \in \mathcal{N}^q$ and $K \in \mathcal{N}^s$. (In all other cases \circ_2 holds by virtue of holding for q or s .)

Since $Z^s = Z^q$ there are $M' \in \mathcal{N}^s$ and $K' \in \mathcal{N}^q$ such that $i^M = i^{M'}$ and $i^K = i^{K'}$ and, as $s, q \in \mathbb{P}$ we have that \circ_2 holds for each of the two pairs K, M' and K', M . Thus, if $i^K \leq i^M$ we have that $X_{i^K}^q = X_{i^K}^s$ implies that \circ_2 holds for K, M , while if $i^K \geq i^M$ then $X_{i^M}^q = X_{i^M}^s$ implies that \circ_2 holds for K, M . ▲5.4

Thus we have shown Claim (5.3) ▲5.3

Hence forcing with \mathbb{P} preserves λ . ▲5.2

Proposition 5.5. *If $2^\kappa \leq \lambda$ then cardinals above λ are preserved.*

Proof. As $2^\kappa \leq \lambda$ the forcing has the λ^+ chain condition. ▲5.5

Of course, if λ is strongly inaccessible then clearly the forcing \mathbb{P} has the λ -chain condition. However, as mentioned above in the preamble to Proposition (5.2), we can in fact show that \mathbb{P} always has the λ -chain condition.

Proposition 5.6. *If $2^\kappa < \lambda$ then \mathbb{P} has the λ -chain condition.*

Proof. Suppose that \mathcal{A} is an antichain in \mathbb{P} and $\overline{\mathcal{A}} = \lambda$. Let χ be large enough so that $\mathcal{A}, \mathbb{P} \in H_\chi$ (e.g., $\chi = (2^\lambda)^+$). Let $N \prec H_\chi$ be such that $\delta^N = N \cap \lambda \in \lambda$ and $\text{cf}(\delta^N) \geq \kappa^+$, $\overline{N} < \lambda$, and $N^\kappa \subseteq N$ and $\mathcal{A}, \mathbb{P} \in N$. Choose $q \in \mathcal{A} \setminus N$.

Claim 5.7. *There is some $s \in \mathcal{A} \cap N$ such that s and q are compatible.*

Proof. As $\theta_{\zeta^q}^q < \kappa^+$ we have that $F^q \text{``}\theta_{\zeta^q}^q$ is of size at most κ , so since $N^\kappa \subseteq N$ we have that $Y = N \cap F^q \text{``}\theta_{\zeta^q}^q \in N$. Also, as $N \cap \lambda \in \lambda$ we have that Y is an initial segment of $F^q \text{``}\theta_{\zeta^q}^q$. As $N^\kappa \subseteq N$ we also have that each $X_i^q \in N$.

Let $\phi(x)$ be the formula which is the conjunction of the following sub-formulae: $x \in \mathbb{P}$; $x \in \mathcal{A}$; $S^x = S^q$; Y is an initial segment of $F^x \text{``}\theta_{\zeta^x}^x$; $Z^x = Z^q$; and $\langle X_i^x \mid i \in Z^x \rangle = \langle X_i^q \mid i \in Z^q \rangle$.

As q witnesses $H_\chi \models \exists x \phi(x)$ there is some $s \in N$ such that $N \models \phi(s)$.

Define r exactly as in the proof of Claim (5.3): by setting $\zeta^r = \zeta^q + 1$, $S^r \restriction \zeta^r = S^q = S^s$, $\theta_{\zeta^r}^r = \text{otp}(\text{rge}(F^s) \cup \text{rge}(F^q)) + 1$, $\mathcal{F}_{\zeta^q \zeta^r}^r = \{\text{id}, h_{\zeta^q}^r\}$, where the splitting point of $\mathcal{F}_{\zeta^q \zeta^r}^r$, $\sigma_{\zeta^q}^r$, is $\text{otp}(Y)$, $\mathcal{F}_{i \zeta^r}^r = \{f \cdot g \mid f \in \mathcal{F}_{\zeta^q \zeta^r}^r \text{ \& } g \in \mathcal{F}_{i \zeta^q}^r\}$ for $i < \zeta^q$, $F^r \restriction \text{otp}(\text{rge}(F^s) \cup \text{rge}(F^q))$ is the inverse of the transitive collapse of $\text{rge}(F^s) \cup \text{rge}(F^q)$, $F^r(\text{otp}(\text{rge}(F^s) \cup \text{rge}(F^q))) = \text{ssup}(\text{rge}(F^s) \cup \text{rge}(F^q))$, and $\mathcal{N}^r = \mathcal{N}^s \cup \mathcal{N}^q$.

Subclaim 5.8. *\mathcal{N}^r is such that $r \in \mathbb{P}$, whence $r \in \mathbb{P}$ and $r \leq q, s$.*

Proof. Verbatim as in the proof of Subclaim (5.4). ▲_{5.8}

Thus we have shown there is some $s \in \mathcal{A} \cap N$ compatible with q . ▲_{5.7}

This contradicts the assumption that \mathcal{A} is an antichain. Thus we have shown that \mathbb{P} contains no antichains of size λ . ▲_{5.6}

6. PRESERVATION OF κ AND ALL SMALLER CARDINALS

In this section we show that if κ is a regular cardinal then \mathbb{P} is $<\kappa$ -closed. The proof is similar to the proof in the following section that forcing with \mathbb{P} adds a $(\kappa^+, 1)$ -simplified morass.

Suppose that κ is a regular cardinal.

Proposition 6.1. *The forcing with \mathbb{P} is $<\kappa$ -closed.*

Proof. Let μ be a cardinal with $\mu < \kappa$. Let $\langle p_\alpha \mid \alpha < \mu \rangle$ be a descending sequence of conditions from \mathbb{P} . For $\alpha < \beta < \mu$ let $k_{\alpha\beta}$ witness that $p_\beta \leq p_\alpha$. Note that if $\alpha < \beta < \gamma < \mu$ then $k_{\alpha\gamma} = k_{\beta\gamma} \cdot k_{\alpha\beta}$.

We start by constructing a small SMS segment S into which each S^{p_α} is embedded.

Define two relations \sim and $<$ on the set $\{(\alpha, i) \mid \alpha < \mu \text{ \& } i \leq \zeta^{p_\alpha}\}$, by $(\alpha, i) \sim (\beta, j)$ if $\alpha \leq \beta$ and $k_{\alpha\beta}(i) = j$ or $\beta \leq \alpha$ and $k_{\beta\alpha}(j) = i$, and, similarly, $(\alpha, i) < (\beta, j)$ and $\alpha \leq \beta$ and $k(i) < j$ or $\beta \leq \alpha$ and $k(j) < i$. Clearly \sim is an equivalence relation and $<$ is a linear ordering which respects \sim , inducing a linear ordering on the \sim -equivalence classes $\mathcal{E} = \{[(\alpha, i)]_\sim \mid \alpha < \mu \text{ \& } i \leq \zeta^{p_\alpha}\}$.

For $x = [(\alpha, i)]_\sim \in \mathcal{E}$ let $\theta_x = \theta_i^{p_\alpha}$. Note, by the definition of \sim and of $\leq_{\mathbb{P}}$, that θ_x is independent of the choice of the pair (α, i) . Note, also, that $x = [(\alpha, i)]_\sim < y = [(\beta, j)]_\sim$ if and only if $\theta_x < \theta_y < \kappa^+$. Consequently, $<$ well-orders \mathcal{E} .

For $x, y \in \mathcal{E}$ define $\mathcal{F}_{xy} = \bigcup \{f \in \mathcal{F}_{ij}^{p_\alpha} \mid \alpha < \mu \text{ \& } x = [(\alpha, i)]_\sim \text{ \& } y = [(\alpha, j)]_\sim\}$. Let $\mathcal{F}_x = \{F^{p_\alpha} \cdot f \mid x = [(\alpha, i)]_\sim \text{ \& } f \in \mathcal{F}_{i\zeta_\alpha}^{p_\alpha}\}$.

Let $\{x_\xi \mid \xi < \zeta'\}$ be the $<$ -increasing ordering of \mathcal{E} . Note that as $\overline{\{(\alpha, i) \mid \alpha < \mu \text{ \& } i \leq \zeta^{p_\alpha}\}} < \kappa$ we must have that $\zeta' < \kappa$.

If ζ' is a successor, *i.e.*, there is a $<$ -maximal element of \mathcal{E} , then set ζ to be the predecessor of ζ' . Otherwise, let $\zeta = \zeta'$.

For $\alpha < \mu$ define $k_\alpha : \zeta^{p_\alpha} + 1 \longrightarrow \zeta + 1$ by $k_\alpha(i) =$ that ξ such that $x_\xi = [(\alpha, i)]_\sim$ for $i < \zeta^{p_\alpha}$.

For $\xi < \zeta'$ let $\theta_\xi = \theta_{x_\xi}$. For $\nu \leq \xi < \zeta'$ let $\mathcal{F}_{\nu\xi} = \mathcal{F}_{x_\nu x_\xi}$. If $\zeta' = \zeta + 1$ and $x_\zeta = [(\alpha, i)]_\sim$ for $\alpha < \mu$ and $i = \zeta^{p_\alpha}$, let $F = F^{p_\alpha}$.

If $\zeta = \zeta'$ let $\theta_\zeta = \text{otp}(\bigcup \{\text{rge}(F^{p_\alpha}) \mid \alpha < \mu\})$, let F be the inverse of the transitive collapse of $\bigcup \{\text{rge}(F^{p_\alpha}) \mid \alpha < \mu\}$ and for $\xi < \zeta$ let $\mathcal{F}_{\xi\zeta} = \bigcup \{F^{-1} \cdot F^{p_\alpha} \cdot f \mid \alpha < \mu \text{ \& } k_\alpha(i) = \xi \text{ \& } f \in \mathcal{F}_{i\zeta_\alpha}^{p_\alpha}\}$.

We shall show that the structure $\langle \langle \theta_\xi \mid \xi \leq \zeta \rangle, \langle \mathcal{F}_{\nu\xi} \mid \nu \leq \xi \leq \zeta \rangle \rangle$ has almost all of the properties of a small SMS and then show how to gently interleave limit levels of small cofinality where necessary.

We start with the structure of the sets $\mathcal{F}_{\xi\xi+1}$ for $\xi < \zeta$.

Given $\xi < \zeta$ let $\alpha < \mu$ be such that for some $i < \zeta^{p_\alpha}$ one has $x_\xi = [(\alpha, i)]_\sim$ and $x_{\xi+1} = [(\alpha, i+1)]_\sim$. Then, by the definition of \mathbb{P} , $\mathcal{F}_{ii+1}^{p_\alpha}$ is either a singleton (consisting of a order-preserving map) or an amalgamation pair, and for all $\beta \in (\alpha, \mu)$ we have $k_{\alpha\beta}(i+1) = k_{\alpha\beta}(i) + 1$ and $\mathcal{F}_{k_{\alpha\beta}(i)k_{\alpha\beta}(i)+1}^{p_\beta} = \mathcal{F}_{ii+1}^{p_\alpha}$. Hence we must then have that $\mathcal{F}_{\xi\xi+1} = \mathcal{F}_{ii+1}^{p_\alpha}$ and is a singleton (consisting of a order-preserving map) or an amalgamation pair. (Note that, by the definition of small SMS, if $\mathcal{F}_{\xi\xi+1} = \mathcal{F}_{ii+1}^{p_\alpha} = \{f_i\}$ is a singleton then f_i is not cofinal.)

Now we show the $\mathcal{F}_{\nu\xi}$ have the required factorization property.

Suppose $\nu \leq \tau \leq \xi < \zeta'$ and $f \in \mathcal{F}_{\nu\tau}$ and $g \in \mathcal{F}_{\tau\xi}$. As μ is a limit ordinal there is some $\alpha < \mu$ with some $i \leq j \leq l \leq \zeta^{p_\alpha}$ such that $k_\alpha(i) = \nu$, $k_\alpha(j) = \tau$ and $k_\alpha(l) = \xi$ (i.e., $x_\nu = [(\alpha, i)]_\sim$, $x_\tau = [(\alpha, j)]_\sim$ and $x_\xi = [(\alpha, l)]_\sim$), $f \in \mathcal{F}_{ij}^{p_\alpha}$ and $g \in \mathcal{F}_{jl}^{p_\alpha}$. Thus $g \cdot f \in \mathcal{F}_{il}^{p_\alpha}$, and hence $g \cdot f \in \mathcal{F}_{\nu\xi}$.

Similarly, suppose $\nu \leq \tau \leq \xi < \zeta$ and $h \in \mathcal{F}_{\nu\xi}$. There is some $\alpha < \mu$ and $i \leq j \leq l \leq \zeta^{p_\alpha}$ such that $k_\alpha(i) = \nu$, $k_\alpha(j) = \tau$, $k_\alpha(l) = \xi$ and $h \in \mathcal{F}_{il}^{p_\alpha}$. But then, as S^{p_α} is a small SMS, there are $g \in \mathcal{F}_{jl}^{p_\alpha} \subseteq \mathcal{F}_{\tau\xi}$, and $f \in \mathcal{F}_{ij}^{p_\alpha} \subseteq \mathcal{F}_{\nu\tau}$, such that $g \cdot f = h$.

Now suppose that $\zeta = \zeta'$ and $\nu \leq \tau < \zeta$. If $f \in \mathcal{F}_{\nu\tau}$ and $g \in \mathcal{F}_{\tau\zeta}$ there is some $\alpha < \mu$, $i \leq j \leq l \leq \zeta^{p_\alpha}$ and $h \in \mathcal{F}_{j\zeta^{p_\alpha}}^{p_\alpha}$ such that $k_\alpha(i) = \nu$, $k_\alpha(j) = \tau$, $f \in \mathcal{F}_{ij}^{p_\alpha}$ and $g = F^{-1} \cdot F^{p_\alpha} \cdot h$. Then $g \cdot f = F^{-1} \cdot F^{p_\alpha} \cdot (h \cdot f)$, where $h \cdot f \in \mathcal{F}_{i\zeta^{p_\alpha}}^{p_\alpha} \subseteq \mathcal{F}_{\nu\tau}$, and hence $g \cdot f \in \mathcal{F}_{\nu\zeta}$.

Similarly, if $\zeta = \zeta'$, $\nu \leq \tau < \zeta$ and $f \in \mathcal{F}_{\nu\zeta}$ there is some $\alpha < \mu$, $i \leq j \leq l \leq \zeta^{p_\alpha}$, $h \in \mathcal{F}_{ij}^{p_\alpha}$ and $g \in \mathcal{F}_{j\zeta^{p_\alpha}}^{p_\alpha}$ such that $k_\alpha(i) = \nu$, $k_\alpha(j) = \tau$, and $f = (F^{-1} \cdot F^{p_\alpha} \cdot g) \cdot h$.

Next, we show directedness at limits.

Suppose $\nu, \tau < \xi < \zeta'$, ξ is a limit ordinal, $f \in \mathcal{F}_{\nu\xi}$ and $g \in \mathcal{F}_{\tau\xi}$. Let $\alpha < \mu$ be such that there are $i \leq j < l \leq \zeta^{p_\alpha}$ with $k_\alpha(i) = \nu$, $k_\alpha(j) = \tau$, $f \in \mathcal{F}_{il}^{p_\alpha}$ and $g \in \mathcal{F}_{jl}^{p_\alpha}$.

If l is a successor then $\mathcal{F}_{l-1l}^{p_\alpha}$ is a singleton, say $\{h_{l-1}\}$ (as otherwise $k_{\alpha\beta}(l) = k_{\alpha\beta}(l-1) + 1$ for all $\beta \in (\alpha, \mu)$ and hence $k_\alpha(l) = \xi$ is a successor). Furthermore, as S^p is a SMS there are $f' \in \mathcal{F}_{il-1}^{p_\alpha}$ and

$g' \in \mathcal{F}_{jl-1}^{p_\alpha}$ such that $f = h_{l-1} \cdot f'$ and $g = h_{l-1} \cdot g'$. But then $k_\alpha(l-1) \in [\nu \cup \tau, \xi)$ and $f' \in \mathcal{F}_{\nu k_\alpha(l-1)}$, $g' \in \mathcal{F}_{\tau k_\alpha(l-1)}$ and $h_{l-1} \in \mathcal{F}_{k_\alpha(l-1)\xi}$.

If l is a limit ordinal, by the definition of small SMS, there is some $m \in [\max(\{i, j\}), l)$, and $h \in \mathcal{F}_{ml}^{p_\alpha}$, $f' \in \mathcal{F}_{im}^{p_\alpha}$ and $g' \in \mathcal{F}_{jm}^{p_\alpha}$ such that $f = h \cdot f'$ and $g = h \cdot g'$. Hence $f' \in \mathcal{F}_{\nu k_\alpha(m)}$, $g' \in \mathcal{F}_{\tau k_\alpha(l-1)}$ and $h \in \mathcal{F}_{k_\alpha(m)\xi}$.

Now suppose that $\nu, \tau < \zeta = \zeta'$, $f \in \mathcal{F}_{\nu\zeta}$ and $g \in \mathcal{F}_{\tau\zeta}$. Then there is some $\alpha < \mu$ and $i \leq j \leq \zeta^{p_\alpha}$ with $k_\alpha(i) = \nu$, $k_\alpha(j) = \tau$, and $k_\alpha(\zeta^{p_\alpha}) = \xi < \zeta$, and some maps $f' \in \mathcal{F}_{i\zeta^{p_\alpha}}$ and $g' \in \mathcal{F}_{j\zeta^{p_\alpha}}$ with $f = F^{-1} \cdot F^{p_\alpha} \cdot f'$ and $g = F^{-1} \cdot F^{p_\alpha} \cdot g'$. As $F^{-1} \cdot F^{p_\alpha} \in \mathcal{F}_{\xi\zeta}$ we are done.

We have now shown that $\langle \langle \theta_\xi \mid \xi \leq \zeta \rangle, \langle \mathcal{F}_{\nu\xi} \mid \nu \leq \xi \leq \zeta \rangle \rangle$ has all of the properties of a small SMS, except possibly the property that if $\xi \leq \zeta$ is a limit ordinal then $\text{cf}(\theta_\xi) < \kappa$. In order to guarantee this last property, in general, we must add additional levels to \mathcal{E} .

In order to see why consider the example of some $\langle \alpha_l \mid l < \omega \rangle$ in which for all $l < \omega$ we have $k_{\alpha_l}(i) = i$ for all $i < l$ but $\omega = k_{\alpha_l}(l)$ and $\text{cf}(\theta_l^{p_{\alpha_l}}) = \kappa$. We then need to insert a ‘new’ ω -th level in order to ensure that the θ of the ω -th level of the lower bound we are constructing does not have cofinality κ .

This interleaving of additional levels is not hard to do, but writing down the details formally, as we do in the following four paragraphs, is a little involved.

So, set $\zeta^* = \zeta + 1$ if $\text{cf}(\theta_{x_\zeta}) = \kappa$, and otherwise let $\zeta^* = \zeta$. For $\alpha < \mu$ we define $k_\alpha^* : \zeta^{p_\alpha} \rightarrow \zeta^*$ as follows. For each $i \leq \zeta^{p_\alpha}$ let ξ_i be the greatest limit ordinal such that $\xi_i \leq k_\alpha(i)$ if it exists and undefined otherwise. If ξ_i is defined and $\text{cf}(\theta_{\xi_i}) = \kappa$ then $k_\alpha^*(i) = k_\alpha(i) + 1$. Otherwise we take $k_\alpha^*(i) = k_\alpha(i)$.

For $\xi \leq \zeta^*$ such that $\xi \in \bigcup \{\text{rge } k_\alpha^* \mid \alpha < \mu\}$ we let $\theta_\xi^* = \theta_j^{p_\alpha}$ for any pair (α, j) such that $k_\alpha^*(j) = \xi$. If $\nu \leq \xi \leq \zeta^*$ and $\nu, \xi \in \bigcup \{\text{rge } k_\alpha^* \mid \alpha < \mu\}$ let $\alpha < \mu$ be such that $\nu, \xi \in \text{rge } k_\alpha^*$, let $\nu' = k_\alpha((k_\alpha^*)^{-1}(\nu))$ and $\xi' = k_\alpha((k_\alpha^*)^{-1}(\xi))$ and set $\mathcal{F}_{\nu\xi}^* = \mathcal{F}_{\nu'\xi'}$.

Now suppose otherwise, that $\xi \leq \zeta^*$ but $\xi \notin \bigcup \{\text{rge } k_\alpha^* \mid \alpha < \mu\}$, hence ξ is a limit ordinal. Let α^* be least such that $\xi \in \text{rge}(k_{\alpha^*})$. For all $\alpha \in [\alpha^*, \mu)$ and all $f \in \mathcal{F}_{ij}^{p_\alpha}$, where $k_\alpha(j) = \xi$ let $\gamma_f = \text{ssup}(\text{rge}(f))$. Lemma (3.2) shows that for all such f we have $\gamma_f < \theta_\xi$. Hence, as $\text{cf}(\theta_\xi) = \kappa$, we have $\text{ssup}(\{\gamma_f \mid \alpha \in [\alpha^*, \mu) \ \& \ k_\alpha(j) = \xi \ \& \ f \in \mathcal{F}_{ij}^{p_\alpha}\}) < \theta_\xi$.

So we can choose θ_ξ^* such that $\text{ssup}(\{\gamma_f \mid \alpha \in [\alpha^*, \mu) \& k_\alpha(j) = \xi \& f \in \mathcal{F}_{ij}^{p_\alpha}\}) < \theta_\xi^* < \theta_\xi$ and $\text{cf}(\theta_\xi^*) < \kappa$. We set $\mathcal{F}_{\xi\xi+1}^* = \{\text{id}\}$ and define the other sets of maps in the obvious way using composition to ensure the last clause in the definition of small SMS holds: *e.g.*, for $\nu \in \bigcup\{\text{rge } k_\alpha^* \mid \alpha < \mu\}$ if $\nu < \xi$ we take $\mathcal{F}_{\nu\xi+1}^* = \{\text{id} \cdot f \mid f \in \mathcal{F}_{\nu\xi}^*\}$ and if $\xi < \nu$ we take $\mathcal{F}_{\xi\nu}^* = \{f \cdot \text{id} \mid f \in \mathcal{F}_{\xi+1\nu}^*\}$.

Now set $S = \langle \langle \theta_\xi^* \mid \xi \leq \zeta^* \rangle, \langle \mathcal{F}_{\nu\xi}^* \mid \nu \leq \xi \leq \zeta^* \rangle \rangle$. It is clear that S is an small SMS.

We also define $\mathcal{N} = \{N \mid \exists \alpha < \mu \ N \in \mathcal{N}^{p_\alpha}\}$ and set $p = (S, F, \mathcal{N})$.

In order to finish the proof we must show that $p \in \mathbb{P}$ and that $p \leq p_\alpha$ for each $\alpha < \mu$.

For the former all that is left to show is that \mathcal{N} is as required.

Suppose $M \in \mathcal{N}$. Let $\alpha < \mu$ be such that $M \in \mathcal{N}^{p_\alpha}$, and let $i^M \leq \zeta^{p_\alpha}$ and $f \in \mathcal{F}_{i\zeta^{p_\alpha}}^{p_\alpha}$ witness this. Then M fits $F^{p_\alpha} \cdot f$, $= F \cdot (F^{-1} \cdot F^{p_\alpha} \cdot f)$ and the uniqueness of $(k_\alpha^*(i^M), F^{-1} \cdot F^{p_\alpha} \cdot f)$ follows from Lemma (3.2).

Suppose $i' \leq j' \leq k_\alpha^*(i^M)$ and $f \in \mathcal{F}_{i'j'}$. If $i' \in \bigcup\{\text{rge}(k_\beta^*) \mid \beta \in (\alpha, \mu)\}$ let $i = i'$ and otherwise let $i = i' + 1$. Define j similarly. Then there is some $\beta \in (\alpha, \mu)$ and $\bar{i}, \bar{j} \leq \zeta^{p_\beta}$ such that $k(\bar{i}) = i$, $k(\bar{j}) = j$ and $f \in \mathcal{F}_{\bar{i}\bar{j}}^{p_\beta}$. As $M \in \mathcal{N}^{p_\beta}$ we have that $f \in \overline{M}$.

Similarly, if $N \in \mathcal{N}$, $\gamma < \delta^N$, $g \in (\theta_{i_N})^\gamma$, $i^N \leq i^M$ and $g \in \overline{N}$ there is some $\beta \in (\alpha, \mu)$ such that $N \in \mathcal{N}^{p_\beta}$ and $i^N \in \text{rge}(k_\beta^*)$ and hence $g \in \overline{M}$.

Now let $\alpha < \mu$. We check that $p \leq p_\alpha$.

We take the witnessing ' k ' to be k_α^* . The three bulleted properties of k_α^* are immediate from the definition of k_α^* and that of S . We take $f^{p_\alpha p}$ to be $F^{-1} \cdot F^{p_\alpha}$, making it immediate that $F^{p_\alpha} = F \cdot f^{p_\alpha p}$. It is also immediate from the definition of \mathcal{N} that $\mathcal{N}^{p_\alpha} \subseteq \mathcal{N}$.

Finally, if $N \in \mathcal{N}$ is witnessed by $(k_\alpha^*(i), f^{p_\alpha p} \cdot g)$ for some $g \in \mathcal{F}_{i\zeta^{p_\alpha}}^{p_\alpha}$ then we must show that $N \in \mathcal{N}^p$.

Let $\beta \in [\alpha, \mu)$ be such that $N \in \mathcal{N}^{p_\beta}$. By the definition of k_α^* and the fact that $k_\alpha^* = k_\beta^* \cdot k_{\alpha\beta}$, and since $F^{p_\alpha} = F^{p_\beta} \cdot f^{p_\alpha p_\beta}$, we have that this is witnessed by $(k_{\alpha\beta}(i), f^{p_\alpha p_\beta} \cdot g)$. As $p_\beta \leq p_\alpha$ we have (by the final property in the list of requirements defining $\leq_{\mathbb{P}}$) that $N \in \mathcal{N}^{p_\alpha}$ (and (i, g) witnesses that N fits $F^g \cdot g$). ▲_{6.1}

7. SHOWING \mathbb{P} ADDS AN $(\kappa^+, 1)$ -SIMPLIFIED MORASS.

Lemma 7.1. *Let $p \in \mathbb{P}$, $\theta \in (\text{otp}(\text{rge}(F^p) \cup \{\zeta\}), \kappa^+)$ and $\zeta < \lambda$. Then there is some $q \in \mathbb{Q}$ such that $\zeta^q = \zeta^p + 1$, $\theta = \theta_{\zeta^q}^p$ and $\zeta \in \text{rge}(F^q)$.*

Proof. Define q as follows. Let $\zeta^q = \zeta^p + 1$.

If $\zeta \notin \text{rge}(F^p)$ let $\gamma = \text{otp}(\text{rge}(F^p) \cap \zeta)$. Define $f_{\zeta^p} : \theta_{\zeta^p}^p \rightarrow \theta$ by $f_{\zeta^p} \upharpoonright \gamma = \text{id}$ and $f_{\zeta^p}(\gamma + \xi) = \gamma + 1 + \xi$ for $\xi \in \text{otp}(\theta_{\zeta^p}^p \setminus \gamma)$. Define $F^q : \theta \rightarrow \lambda$ by $F^q(\xi) = F^p(\xi)$ for $\xi < \gamma$, $F^q(\gamma) = \zeta$, $F^q(\gamma + 1 + \xi) = F^p(\gamma + \xi)$ for $\xi \in \text{otp}(\theta_{\zeta^p}^p \setminus \gamma)$ and $F^q(\theta_{\zeta^p}^p + \xi) = \text{ssup}(F^p \restriction \theta_{\zeta^p}^p) + \xi$ for $\xi \in \text{otp}(\theta \setminus (\gamma + 1 + \text{otp}(\theta_{\zeta^p}^p \setminus \gamma)))$.

If, on the other hand, $\zeta \in \text{rge}(F^p)$ let $f_{\zeta^p} = \text{id}$ and let $F^q : \theta \rightarrow \lambda$ be given by $F^q(\xi) = F^p(\xi)$ for $\xi < \theta_{\zeta^p}^p$ and $F^q(\theta_{\zeta^p}^p + \xi) = \text{ssup}(F^p \restriction \theta_{\zeta^p}^p) + \xi$ for $\xi \in \text{otp}(\theta \setminus \theta_{\zeta^p}^p)$.

Finally, let

$$\begin{aligned} S^q &= \langle \langle \theta_i^p \mid i \leq \zeta^p \rangle \cap \theta, \langle \mathcal{F}_{ij}^p \mid i \leq j \leq \zeta^p \rangle \cap \langle \{f_{\zeta^p} \cdot f \mid f \in \mathcal{F}_{i\zeta^p}^p\} \mid i \leq \zeta^p \rangle \rangle, \\ \mathcal{F}^* &= \{(F^q \cdot f, \theta_i^q) \mid i \leq \zeta^p + 1 \text{ \& } f \in \mathcal{F}_{i\zeta^p+1}^q\} \text{ and } \mathcal{N}^q = \mathcal{N}^p. \end{aligned} \quad \blacktriangle_{7.1}$$

Proposition 7.2. *Forcing with \mathbb{P} adds an $(\kappa^+, 1)$ -simplified morass.*

Proof. Let G be \mathbb{P} -generic. Define two relations \sim and $<$ on the set $\{(p, i) \mid p \in G \text{ \& } i \leq \zeta^p\}$, by $(p, i) \sim (q, j)$ if there is some $r \in G$ with $r \leq p, q$ witnessed by $k : \zeta^p \rightarrow \zeta^r$ and $l : \zeta^q \rightarrow \zeta^r$ and $k(i) = l(j)$ and, similarly, $(p, i) < (q, j)$ if there is some $r \in G$ with $r \leq p, q$ witnessed by $k : \zeta^p \rightarrow \zeta^r$ and $l : \zeta^q \rightarrow \zeta^r$ and $k(i) < l(j)$. Clearly \sim is an equivalence relation and $<$ is a linear ordering which respects \sim , inducing a linear ordering on the \sim -equivalence classes $\mathcal{E} = \{[(p, i)]_\sim \mid p \in G \text{ \& } i \leq \zeta^p\}$.

For $x = [(p, i)]_\sim \in \mathcal{E}$ let $\theta_x = \theta_i^p$. Note, by the definition of \sim and of $\leq_{\mathbb{P}}$, that θ_x is independent of the choice of the pair (p, i) . Note, also, that $x = [(p, i)]_\sim < y = [(q, j)]_\sim$ if and only if $\theta_x < \theta_y < \kappa^+$. Consequently, $<$ well-orders \mathcal{E} and, by Lemma (7.1), it does so in order-type κ^+ , since $\{\theta_x \mid x \in \mathcal{E}\}$ is unbounded in κ^+ . Let $\{x_\alpha \mid \alpha < \kappa^+\}$ be the $<$ -increasing ordering of \mathcal{E} .

For $x, y \in \mathcal{E}$ define $\mathcal{F}_{xy} = \bigcup \{f \in \mathcal{F}_{ij}^p \mid p \in G \text{ \& } x = [(p, i)]_\sim \text{ \& } y = [(p, j)]_\sim\}$. Let $\mathcal{F}_x = \{F^p \cdot f \mid p \in G \text{ \& } x = [(p, i)]_\sim \text{ \& } f \in \mathcal{F}_{i\zeta^p}^p\}$.

For $\alpha < \kappa^+$ let $\theta_\alpha = \theta_{x_\alpha}$. Let $\theta_{\kappa^+} = \lambda$. For $\alpha \leq \beta < \kappa^+$ let $\mathcal{F}_{\alpha\beta} = \mathcal{F}_{x_\alpha x_\beta}$. For $\alpha < \kappa^+$ let $\mathcal{F}_{\alpha\kappa^+} = \mathcal{F}_{x_\alpha}$.

It remains to check that $M = \langle \langle \theta_\alpha \mid \alpha \leq \kappa^+ \rangle, \langle \mathcal{F}_{\alpha\beta} \mid \alpha \leq \beta \leq \kappa^+ \rangle \rangle$ is an $(\kappa^+, 1)$ -simplified morass.

Given $\alpha < \kappa^+$ let $p \in G$ be such that for some $i < \zeta^p$ one has $x_\alpha = [(p, i)]_\sim$ and $x_{\alpha+1} = [(p, i+1)]_\sim$. Then, by the definition of \mathbb{P} , \mathcal{F}_{ii+1}^p is either a singleton (consisting of a order-preserving map) or an amalgamation pair, and by the definition of $\leq_{\mathbb{P}}$, one must then have that $\mathcal{F}_{\alpha\alpha+1}$ is a singleton (consisting of a order-preserving map) or an amalgamation pair.

Suppose $\alpha \leq \beta \leq \gamma \leq \kappa^+$ and $f \in \mathcal{F}_{\alpha\beta}$ and $g \in \mathcal{F}_{\beta\gamma}$. If $\gamma < \kappa^+$ then by the directedness of G there is some $p \in G$ and $i \leq j \leq k \leq \zeta^p$ such that $x_\alpha = [(p, i)]_\sim$, $x_\beta = [(p, j)]_\sim$, $x_\gamma = [(p, k)]_\sim$, $f \in \mathcal{F}_{ij}^p$ and $g \in \mathcal{F}_{jk}^p$. Thus $g \cdot f \in \mathcal{F}_{ik}^p$, and hence $g \cdot f \in \mathcal{F}_{\alpha\gamma}$. Similarly, if $\gamma = \kappa^+$ there is some $p \in G$ and $i \leq j \leq \zeta^p$ such that $x_\alpha = [(p, i)]_\sim$, $x_\beta = [(p, j)]_\sim$, $f \in \mathcal{F}_{ij}^p$ and $g = F^p \cdot g'$ where $g' \in \mathcal{F}_{j\zeta^p}^p$. Thus $g \cdot f = F^p \cdot g' \cdot f \in \mathcal{F}_{\alpha\kappa^+}$.

Similarly, suppose $\alpha \leq \beta \leq \gamma \leq \kappa^+$ and $h \in \mathcal{F}_{\alpha\gamma}$. If $\gamma < \kappa^+$ there is some $p \in G$ and $i \leq j \leq k \leq \zeta^p$ such that $x_\alpha = [(p, i)]_\sim$, $x_\beta = [(p, j)]_\sim$, $x_\gamma = [(p, k)]_\sim$ and $h \in \mathcal{F}_{ik}^p$. But then, as S^p is a small SMS, there are $g \in \mathcal{F}_{jk}^p \subseteq \mathcal{F}_{\beta\gamma}$, and $f \in \mathcal{F}_{ij}^p \subseteq \mathcal{F}_{\alpha\beta}$, such that $g \cdot f = h$. Likewise, if $\gamma = \kappa^+$ there is some $p \in G$ and $i \leq j \leq \zeta^p$ such that $x_\alpha = [(p, i)]_\sim$, $x_\beta = [(p, j)]_\sim$, and $h = F^p \cdot h'$ with $h' \in \mathcal{F}_{i\zeta^p}^p$. Then, once more, as S^p is a small SMS, there are $g \in \mathcal{F}_{j\zeta^p}^p$ and $f \in \mathcal{F}_{ij}^p \subseteq \mathcal{F}_{\alpha\beta}$, such that $g \cdot f = h'$, and so $F^p \cdot g \in \mathcal{F}_{\beta\kappa^+}$.

Directedness at limits: suppose $\alpha \leq \beta < \varepsilon \leq \kappa^+$, $f \in \mathcal{F}_{\alpha\varepsilon}$ and $g \in \mathcal{F}_{\beta\varepsilon}$, and ε is a limit ordinal.

First of all suppose $\varepsilon < \kappa^+$. By the directedness of G let $p \in G$ be such that there are $i \leq j < k \leq \zeta^p$ with such that $x_\alpha = [(p, i)]_\sim$, $x_\beta = [(p, j)]_\sim$, $x_\varepsilon = [(p, k)]_\sim$, $f \in \mathcal{F}_{ik}^p$ and $g \in \mathcal{F}_{jk}^p$.

If $\text{cf}(\varepsilon) = \kappa$ then k is a successor and \mathcal{F}_{k-1k}^p is a singleton, say $\{h_{k-1}\}$. Furthermore, as S^p is a SMS there are $f' \in \mathcal{F}_{ik-1}^p$ and $g' \in \mathcal{F}_{jk-1}^p$ such that $f = h_{k-1} \cdot f'$ and $g = h_{k-1} \cdot g'$. But then there is some $\gamma \in [\alpha \cup \beta, \varepsilon)$ such that $x_\gamma = [(p, k-1)]_\sim$ and $f' \in \mathcal{F}_{\alpha\gamma}$, $g' \in \mathcal{F}_{\beta\gamma}$ and $h_{k-1} \in \mathcal{F}_{\gamma\varepsilon}$.

If $\text{cf}(\varepsilon) < \kappa$ we may suppose by the closure of \mathbb{P} and genericity that k is a limit ordinal. Whence by the definition of SMS that there are $l \in [\max(\{i, j\}), k)$, with $x_\gamma = [(p, l)]_\sim$, $h \in \mathcal{F}_{lk}$, $f' \in \mathcal{F}_{il}$ and $g' \in \mathcal{F}_{jl}$

such that $f = h \cdot f'$ and $g = h \cdot g'$. Hence $f' \in \mathcal{F}_{\alpha\gamma}$, $g' \in \mathcal{F}_{\beta\gamma}$ and $h \in \mathcal{F}_{\gamma\varepsilon}$.

If $\varepsilon = \kappa^+$ then, again by the directedness of G , there is some $p \in G$ such that there are $i \leq j < \zeta^p$ such that $x_\alpha = [(p, i)]_\sim$, $x_\beta = [(p, j)]_\sim$, and there are $f' \in \mathcal{F}_{i\zeta^p}^p$ and $g' \in \mathcal{F}_{j\zeta^p}^p$ with $f = F^p \cdot f'$ and $g = F^p \cdot g'$.

Thus we have verified directedness at ε .

We have that $\bigcup \{f \restriction \theta_\alpha \mid \alpha < \kappa^+ \text{ \& } f \in \mathcal{F}_{\alpha\kappa^+}\} = \lambda$ by Lemma (7.1).

Finally, we shall show that $V[G] \models \hat{\lambda} = \dot{\kappa}_2$.

We recall a useful lemma of Velleman's and a couple of related definitions.

Lemma 7.3. ([22], Lemma (3.2).) *If $\alpha \leq \beta \leq \kappa^+$, $f_0, f_1 \in \mathcal{F}_{\alpha\beta}$, $\tau_0, \tau_1 < \theta_\alpha$ and $f_0(\tau_0) = f_1(\tau_1)$ then $\tau_0 = \tau_1$ and $f_0 \restriction \tau_0 + 1 = f_1 \restriction \tau_1 + 1$.*

Proof. By induction on β for each fixed α , using the simplicity of the structure of the sets $\mathcal{F}_{\beta'\beta'+1}$ at successor steps and the directedness at limits at limit steps. ▲7.3

Definition 7.4. ([13]) If $\alpha \leq \beta \leq \kappa^+$, $\tau' < \theta_\alpha$ and there is some $f \in \mathcal{F}_{\alpha\beta}$ and $\tau < \theta_\beta$ such that $f(\tau') = \tau$, let $\psi_{(\alpha, \tau'), (\beta, \tau)} = f \restriction \tau' + 1$.

By Lemma (7.3) each such $\psi_{(\alpha, \tau'), (\beta, \tau)}$ is well defined.

Definition 7.5. ([13]) If $\alpha < \kappa^+$, $\tau < \lambda$ and there is some $f \in \mathcal{F}_{\alpha\kappa^+}$ with $\tau \in \text{rge}(f)$ write τ_α for the unique τ' such that $f(\tau') = \tau$.

Let $\xi < \lambda$. There is some $\alpha_\xi < \kappa^+$ such that for all $\alpha \in [\alpha_\xi, \kappa^+)$ there is some $\xi_\alpha < \theta_\alpha$ with $\psi_{(\alpha, \xi_\alpha), (\kappa^+, \xi)}$ well defined. For such α set $A_\alpha = \text{rge}(\psi_{(\alpha, \xi_\alpha), (\kappa^+, \xi)}) \restriction \xi_\alpha$.

By directedness at the limit ordinal κ^+ , we have that for every $\tau < \xi$ there is some $\alpha \in [\alpha^*, \kappa^+)$ and $f \in \mathcal{F}_{\alpha\kappa^+}$ such that $\tau, \xi \in \text{rge}(f)$, and hence $\tau \in A_\alpha$. So $\xi = \bigcup \{A_\alpha \mid \alpha \in [\alpha_\xi, \kappa^+)\}$ and thus is the union of the κ^+ many sets of size at most κ ; hence we must have that $V[G] \models \hat{\lambda} = \dot{\kappa}_2$. ▲7.2

In [19], Shelah, Väänänen and Veličković introduced the antichain property for simplified morasses, a property related to prior work of Miyamoto ([12]).

Definition 7.6. ([19]) Let $M = \langle \langle \theta_\alpha \mid \alpha \leq \kappa^+ \rangle, \langle \mathcal{F}_{\alpha\beta} \mid \alpha \leq \beta \leq \kappa^+ \rangle \rangle$ be a $(\kappa^+, 1)$ -simplified morass. As in Definition (7.5), if $\alpha < \kappa^+$, $\tau < \kappa^{++}$ and there is some $f \in \mathcal{F}_{\alpha\kappa^+}$ with $\tau \in \text{rge}(f)$ write τ_α for the unique τ' such that $f(\tau') = \tau$. We say M has the κ^{++} -antichain property if

$$\forall X \in [\kappa^{++}]^{\kappa^{++}} \exists \tau, \xi \in X \forall \alpha < \kappa^+ \\ (\tau_\alpha \text{ and } \xi_\alpha \text{ are both defined} \implies \tau_\alpha \leq \xi_\alpha).$$

We note that the analogue for $(\omega, 1)$ -simplified morasses always fails if Martin's Axiom holds: whilst ZFC shows there are always $(\omega, 1)$ -simplified morasses, MA_{ω_1} implies that no $(\omega, 1)$ -simplified morass has the ω_1 -antichain property. As far as we know it is open whether there is a similar consistency result for $(\kappa^+, 1)$ -simplified morasses.

Proposition 7.7. *The $(\kappa^+, 1)$ -simplified morass added by forcing with \mathbb{P} has the κ^{++} -antichain condition.*

Proof. Suppose $p \in \mathbb{P}$ and $p \Vdash \dot{X} \in [\kappa^{++}]^{\kappa^{++}}$. We can find some $E \in [\kappa^{++}]^{\kappa^{++}}$ and for each $\xi \in E$ some $p_\xi \in \mathbb{P}$ such that $p_\xi \Vdash s \in \dot{X}$. By Lemma(7.1) we may assume that $\xi \in \text{rge}(F^{p_\xi})$ for each $\xi \in E$. By the Δ -system lemma we may thin E , if necessary, so that for distinct $\xi, \xi' \in E$ we have $\text{dom}(F^{p_\xi}) = \text{dom}(F^{p_{\xi'}})$, there is some $\bar{\xi}$ such that $F^{p_\xi}(\bar{\xi}) = \xi$, $F^{p_{\xi'}}(\bar{\xi}) = \xi'$, $\xi' \notin \text{rge}(F^{p_\xi})$ and $\xi \notin \text{rge}(F^{p_{\xi'}})$. Now apply the proof of Proposition (5.6) to the collection of p_ξ for ξ in the thinned E . This gives us some $\xi, \xi' \in E$ with $\xi < \xi'$ and some $r \leq p_\xi, p_{\xi'}$ such that $r \Vdash \forall \alpha < \kappa^+ (\xi_\alpha \text{ and } \xi'_\alpha \text{ are both defined} \implies \xi_\alpha \leq \xi'_\alpha)$. $\blacktriangle_{7.7}$

8. SUMMARY OF RESULTS PROVEN

Theorem 8.1. *Suppose λ is regular and $2^\kappa < \lambda$. There is a κ^+ -proper, λ -chain condition forcing (hence preserving κ^+ , λ and all larger cardinals and cofinalities) which collapses any cardinals μ with $\kappa^+ < \mu < \lambda$, and such that in the forcing extension there is an $(\kappa^+, 1)$ -simplified morass with the κ^{++} -antichain condition.*

Proof. Immediate from Propositions (4.1), (5.6), (7.2) and (7.7). \blacktriangle

Corollary 8.2. *If $2^\kappa = \kappa^+$ then there is a κ^+ -proper, κ^{++} -cc forcing which preserves all cardinals and cofinalities and adds an $(\kappa^+, 1)$ -simplified morass with the κ^{++} -antichain condition.*

Proof. By the premise that $2^\kappa = \kappa^+$ we may apply Theorem (8.1) with $\lambda = \kappa^{++}$. $\blacktriangle_{8.2}$

Corollary 8.3. *If $2^\kappa < \lambda$ the forcing \mathbb{P} of Theorem (8.1) adds an κ^+ -Kurepa tree. If $\kappa = \omega$ then \mathbb{P} adds \square_{ω_1} sequence.*

Proof. Immediate from Theorem (8.1) and the work of Velleman in [22], and [22] and [21], respectively, where it is shown that if there is a $(\kappa^+, 1)$ -simplified morass then there is a Kurepa tree, and if there is a $(\omega_1, 1)$ -simplified morass then \square_{ω_1} holds. $\blacktriangle_{8.3}$

Remark 8.4. That, when $\kappa = \omega$, \mathbb{P} adds a Kurepa tree should be contrasted with the old theorem of Jensen that if \square_{ω_1} holds there is a ccc forcing – with finite working parts – to add an ω_1 -Kurepa tree. See [24] for a proof of this result in the style of the introductory remarks to this paper.

However there can be no direct extension of that theorem to a one that asserts that if \square_{ω_1} there is ccc forcing to add an $(\omega_1, 1)$ -simplified morass. This is because \square_{ω_1} is consistent with the weak Chang conjecture holding, no ccc forcing can destroy the truth of the weak Chang conjecture and the existence of an $(\omega_1, 1)$ -simplified morass implies the failure of the weak Chang conjecture.

In future work we shall address higher gap analogues of these results.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE LONDON, GOWER ST.,
LONDON, WC1E 6BT, UK

E-mail address: `charles.morgan@ucl.ac.uk`